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ABSTRACT

This proceedings includes the following papers: (1) "Structure of Attention in Teaching Mathematics" (John Mason); (2) "Communicating Mathematics or Mathematics Storytelling" (Kathy Heinrich); (3) "Assessing Mathematical Thinking" (Florence Glanfield and Pat Rogers); (4) "From Theory to Observational Data (and Back Again)" (Carolyn Kieran and Jo Towers); (5) "Bringing Ethnomathematics into the Classroom in a Meaningful Way" (Kgomotso Garegae-Garekwe); (6) "Mathematical Software for the Undergraduate Curriculum" (Frederic Gourdeau, Michael B. Monagan, and Joel Hillel); (7) "Professional Development for Preservice Mathematics Teachers" (Nadine Bednarz and Linda Gattuso); (8) "How Does the Achievement of Canadian Students Compare to That of Students in Other Countries?" (David Robitaille); (9) "The Commonsense of Teaching" (David Wheeler); (10) "A Look at the Conference Logo" (Klaus Hoechsmann); (11) "Preservice Primary Teachers' Beliefs about Mathematics and Levels of Cognitive Functioning" (Sandra Frid); (12) "Where Do I Want Students' Attention? And What Can I Do To Affect Their Attention?" (Dave Hewitt); (13) "Impact Math: A Mathematics Reform Project for Ontario Grade 7 and 8 Teachers" (Doug McDougall); (14) "A Man Left Albuquerque Heading East: Word Problems as Foundational Narrative in Mathematics Education" (Susan Gerofsky); (15) "Learning Algebra Personally" (Ralph Mason); (16) "Exemplary Mathematics Teachers: Subject Conceptions and Instructional Practices" (Geoffrey Roulet); (17) "Recognizing and Supporting Children's Conceptions of Number" (Heather Kelleher); and (18) "Learning to Teach Prospective Teachers: A Teacher Educator's View" (Cynthia Nicol). (ASK)

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CANADIAN MATHEMATICS EDUCATION STUDY GROUP

GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

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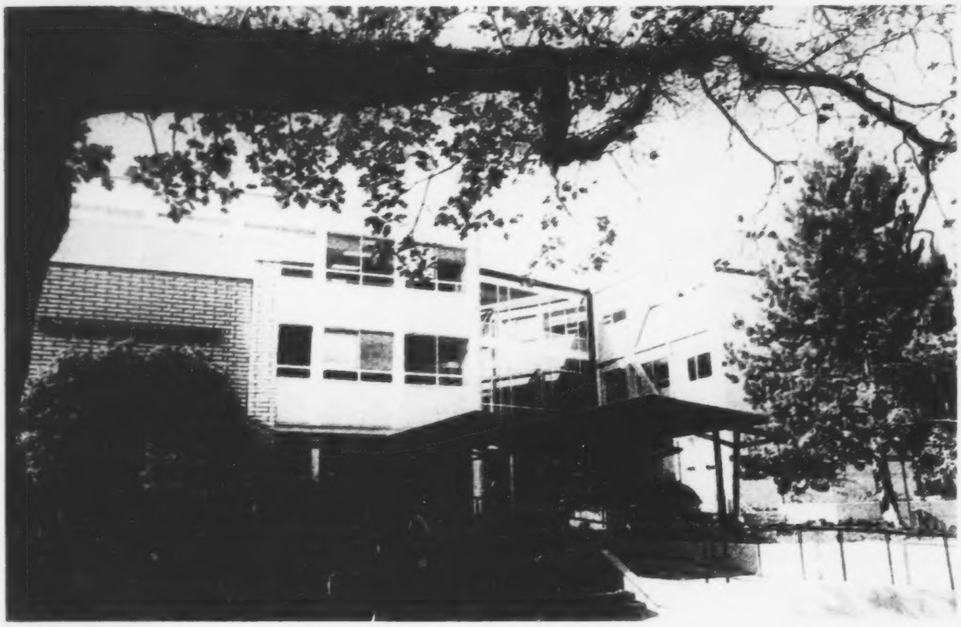

University of British Columbia
May 29-June 2, 1998

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EDITED BY
Yvonne M. Pothier
Mount Saint Vincent University

CANADIAN MATHEMATICS EDUCATION STUDY GROUP

GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

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University of British Columbia, 1998

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EDITOR'S FORWARD

I wish to thank all those who contributed reports for inclusion in these Proceedings. The care taken in preparing a hard copy and disk file of the report, together with camera ready figures, made my work as editor a pleasant task. The value of these Proceedings is entirely the credit of the report authors.

These Proceedings will serve to revive the memories of those who participated in the meeting and hopefully will help generate continued discussion on the varied issues raised during the meeting.

With the completion of these Proceedings and retirement just months away, I bid a fond farewell to all members of CMESG. It has been a pleasure for me to come to know you and I wish each one joy and peace.

Yvonne M. Pothier
Mount Saint Vincent University
August, 1998

ACKNOWLEDGEMENTS

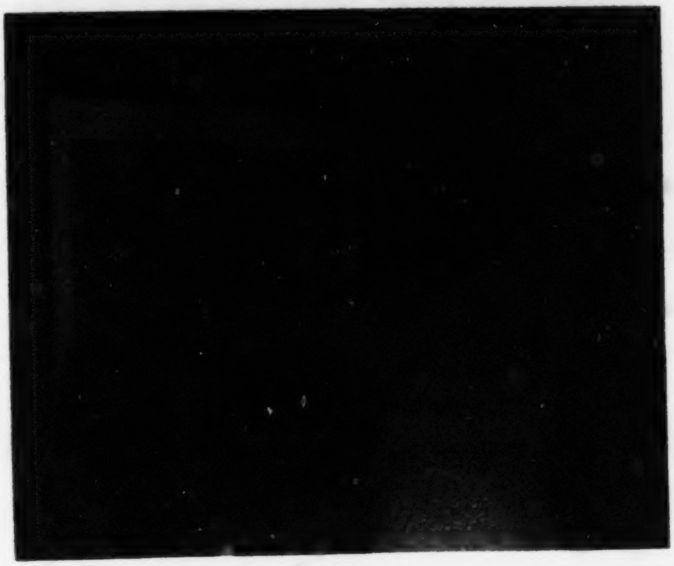
The executive and members would like to thank the University of British Columbia, Vancouver, BC, for hosting the meeting and providing excellent facilities. Special thanks are extended to the local organizing committee, namely, Susan Pirie (chairperson), Ann Anderson, Heather Kelleher, Cynthia Nicol, David Robitaille and James Sherill for their time and work prior to and during the meeting to make the experience pleasant and enjoyable for all participants. The assistance to the committee provided by Saroj Chand and Michelle MacDonald is also acknowledged.

Gratitude is also extended to the guest lecturers,working group leaders, topic session leaders, ad hoc presenters, new PhD graduates and all participants. You are the ones who made the meeting an intellectually stimulating and worthwhile experience: the sunshine and beautiful campus added to the enjoyment.

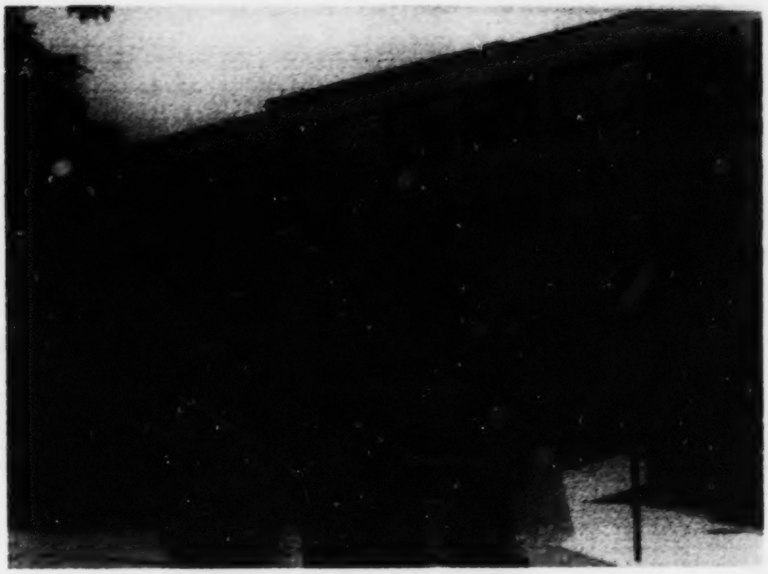
SCHEDULE

Friday May 29	Saturday May 30	Sunday May 31	Monday June 01	Tuesday June 02
	9:00 Working Groups 10:15 Refreshments ------------ 12:15 Lunch	9:00 Working Groups 10:15 Refreshments ------------ 12:15 Lunch	9:00 Working Groups 10:15 Refreshments ------------ 12:15 Lunch	9:00 Plenary II Kathy Heinrich ------------ 10:00 Discussion Plenary II ------------ 10:45 Refreshments 11:15 Working group reports and closing session
------------ 5:30 Opening Plenary General Introductions ------------	1:30 Small group discussion - Plenary I ------------ 2:00 Questions from small groups to speaker ------------ 3:00 Ad hoc Sessions	1:30 Topic Sessions A and B ------------ 2:30 New PhDs (Ad hoc sessions) ------------ 3:30 Refreshments ------------ 4:00 New PhDs ------------ 5:00 Free	1:30 Topic Sessions C and D ------------ 2:30 Ad hoc sessions ------------ 3:30 Refreshments ------------ 4:00 Annual General Meeting	
6:30 Plenary I Ralph Mason ------------ 7:30 Reception	6:30 No host optional group dinner ------------	5:30 Coach to station for train ride and dinner	5:30 Free 6:30 Dinner	

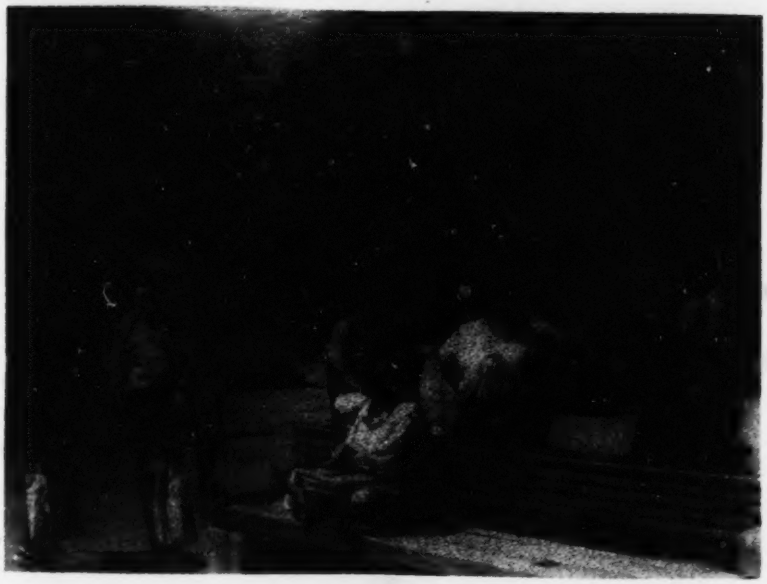
The 1998-99 Executive
Elaine Simmt, Recording Secretary; Frédéric Gourdeau, Membership Secretary,
Malgorzata Dubiel, President; Mary Crowley, Conference Co-coordinator;
Susan Pirie, Vice-President; Eric Muller, Treasurer



Plenary Session Speakers
Kathy Heinrich, John Mason



Free moments to enjoy the beautiful campus ...



... and to chat with colleagues

INTRODUCTION

It is my great pleasure to write an introduction to the fourth and, alas, the last volume of the CMESG/GCEDM proceedings to be edited by Yvonne Pothier. Yvonne has done a wonderful job over the last four years, creating rich, beautiful and very professional-looking chronicles of our meetings. Thank you, Yvonne!

One of the essential requirements of all introductions to the CMESG/GCEDM Proceedings is an attempt to explain to the readers, some of whom may be newcomers to our organization, that the volume in their hands cannot possibly convey the spirit of the meeting it reports on. It can merely describe the content of the activities, without giving much of the flavour of the process. To understand this, one needs to understand the uniqueness of both our organization and our annual meetings.

First, CMESG is an organization unlike other professional organizations. One belongs to it not because of who one is professionnally, but because of one's interests. And that is why our members are members of mathematics and education departments at Canadian and other universities and colleges, and school teachers, united by their interest in mathematics and how it is taught at every level, by the desire to make teaching more exciting, more relevant, more meaningful.

Our meetings are unique, too. One does not simply attend a CMESG meeting the way one attends other professional meetings, by coming to listen to a few chosen talks. You are immediately part of it, you live and breathe it.

The heart of each CMESG meeting is the Working Groups. Participants choose one of several possible topics, and, for three days, became members of a community which meets 3 hours each day to exchange ideas and knowledge, and, through discussions which often continue beyond the allotted time, create fresh knowledge and insights.

Throughout the three days, the group becomes much more than a sum of its parts and, often in ways totally unexpected to its leaders, who, after working for months prior to the meeting, may see their carefully prepared plan ignored or put aside by the group, and a completely new picture emerging in its stead.

Two panel talks are traditionally part of the conference: one given by a mathematician, one by a mathematics educator; at least one is invited from outside Canada, to bring a non-Canadian perspective to these meetings. These speakers participate in the whole meeting; some of them afterwards became part of the Group. And, in the spirit of CMESG meetings, a plenary talk is not just a talk, but a mere beginning: it is followed by discussions in small groups which prepare questions for the speaker. After the small group discussions, in a renewed plenary session, the speaker fields the questions generated by the groups.

Topic Groups and Ad-hoc presentations provide more possibilities for exchange of ideas and reflections. Shorter in duration than the Working Groups, Topic Groups are sessions where individual members present work in progress and often find inspiration and new insight from their coleagues' comments.

Ad-hoc sessions are opportunities to share ideas which are often not even "half-baked" - sometimes born during the very meeting at which they are presented. A traditional part of each meeting is the recognition of new PhD's. *Those who completed their dissertations in the last year are invited to speak* on their work. This gives the group a wonderful opportunity to observe the changing face of mathematics education in Canada.

Our annual meetings are traditionally set in university campuses with participants staying in dormitories rather than hotels, both to make the meetings more affordable and to allow for discussions to continue far beyond the scheduled hours, at times ending in the increasingly famous midnight "pizza runs".

The 1998 Annual Meeting was no exception. Hosted on the beautiful campus of the University of British Columbia with its own beaches, Botanical Garden, Museum of Anthropology and many other attractions, it was a memorable meeting. Even the weather was uncharacteristically warm and sunny for BC.

Malgorzata Dubiel
President (1998-1999)

PLENARY LECTURES

Plenary Lecture 1

LA STRUCTURE DE L'ATTENTION DANS L'ENSEIGNEMENT DES MATHEMATIQUES

STRUCTURE OF ATTENTION IN TEACHING MATHEMATICS

John Mason, Open University, Milton Keynes, UK

INTRODUCTION

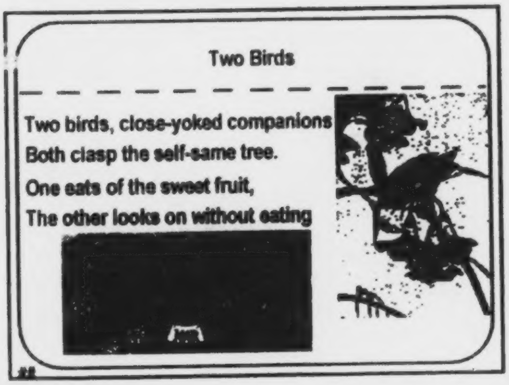
I wish to put before you the conjecture that what your students are attending to is crucial for what they learn from being in your presence. I shall argue that what you are attending to, and the structure of your attention, can profoundly influence students.

I suggest that the structure of students' attention constitutes their awareness and their learning-state. It follows that it is critical that student attention be directed where it needs to be in order to learn efficiently, and in order to know where it needs to be it is valuable to become aware of where my own attention is placed.

In particular, I want to work on different aspects of the structure of attention, which I shall define phenomenologically through experience. Until I really appreciate the nature and the variety of structures associated with the topics I teach, I cannot be fully informed about how to teach them.

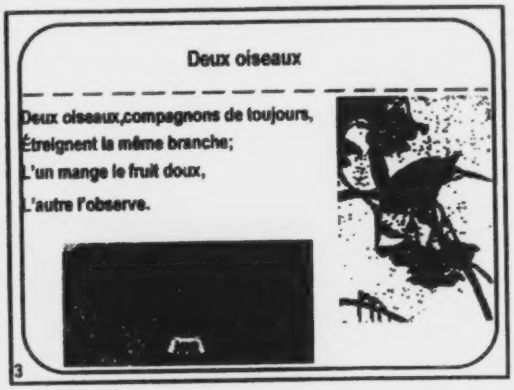
I shall draw on a wide variety of sources, but I want to convince you through your making sense of your experience in this lecture, not through quoting the wisdom of the ancients of our community.

Since we are going to be doing some mathematics together, I must remind you that you will get the most out of these tasks if you work at liberating your observer, your second bird, as described in this stanza from the Rg Veda (quoted by Bennett 1964 p108; see also Zaehner 1966 p190, 210).



Two Birds

Two birds, close-yoked companions
Both clasp the self-same tree.
One eats of the sweet fruit,
The other looks on without eating



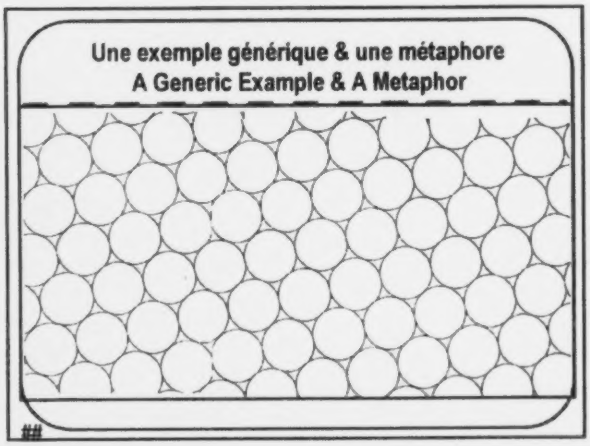
Deux oiseaux

Deux oiseaux,compagnons de toujours,
Étreignent la même branche;
L'un mange le fruit doux,
L'autre l'observe.

As with any powerful image there are multiple interpretations of the two birds. I choose here to see them as aspects of the structure of awareness. One bird eats of the sweet fruit of mathematics, working away at the outer task (Tahta 1980, 1981 distinguishes between outer and inner task or outer and inner meaning of a task), while the other looks on without eating, observing without judging or commenting. It is essential to awaken this inner monitor (Mason et al 1982), this executive (Schoenfeld 1985).

EXPERIENCES OF SHIFTS OF ATTENTION

Let me start with a task which is both metaphoric and generic.



Une exemple générique & une métaphore
A Generic Example & A Metaphor

Look at the main part of the slide and subvocally say to yourself what you are seeing. Pay attention to any technical terms you find yourself using.

I hope you found yourself seeing an infinite configuration of touching circles. Notice how the infinity is perceived: it is clearly a 'sense-of', not a direct perception. In the lecture I invited people to say what they saw to someone else, and to listen to what they had to say. Usually people find that the descriptions are different even if (and it is not a frequent if) they are seeing in exactly the same way. I then directed attention using the following sequence:

1 see the configuration as filling the plane

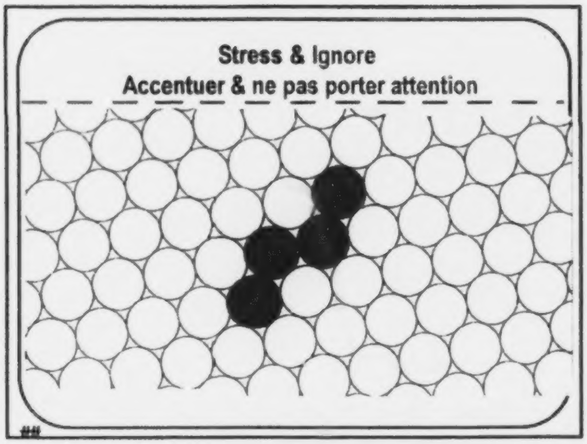
2 see rows of circles, in three different directions

3 see a single circle surrounded by six in a ring, and so on

4 see two adjacent circles surrounded by a ring of circles and so on ...

5 see three circles touching each other and then see them as surrounded by a layer of circles, and so on...

Your responses to each of these directives involve shifts of attention because you find yourself stressing and ignoring differently. I could have used colour to accentuate rings, and then I would have known that you were perceiving ring structures without having to use words. The central point is that you stress certain features and ignore others. Now see the shaded circles as a unit.

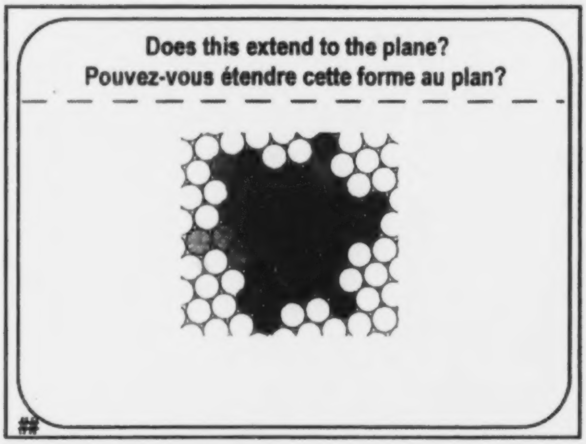


Stress & Ignore
Accentuer & ne pas porter attention

What do you need to do with your attention in order to see that this shape will fill out the tessellation, will tessellate the pattern of circles?

I suggest that the colouring induces a stressing and consequent ignoring. Furthermore you might find yourself with a sense of the particular as generic: the highlighted piece reproducing itself in further copies. Connections are made with ways in which you know will tessellate the figure, either as pairs of layers or as sets of four consecutive layers. Notice that I reinforced the layer-notion at the beginning of the previous slide, but then tried to suppress it with a sequence of ring-structures. Nevertheless the perception of layers dominates the ring-structure. Making yourself change the way you stress is one way to shift the structure of your attention, even against current emphases, but there are many more.

What if I use this unit of four circles to start tessellating in a different way?



Does this extend to the plane?
Pouvez-vous étendre cette forme au plan?

Suddenly you are jolted out of the layers you were seeing.

But it is hard using this configuration to tell whether the tessellation can actually be completed. You probably need to repeat the colouring yourself in order to get a sense of what is going on, hoping that that sense enables you to see how it might extend indefinitely in the same way that the simpler ring structures worked.

This is a good example of the role of specialising in mathematics: the purpose of specialising is not just to collect data towards some generality, but to enable you to attend to how you are developing the particular cases so that you can see through the particular to the general.

I see these tessellations by grouping of circles as a generic task, because it struck me that in common with most pedagogic tasks, when I suddenly see a different way to stress and ignore, I find myself wanting to ask others some version of the question: do you see what I see? But seeing someone else's way is often difficult. Furthermore, as we found, there is a considerable amount of letting go and hanging on, stressing and ignoring, which goes into 'seeing'.

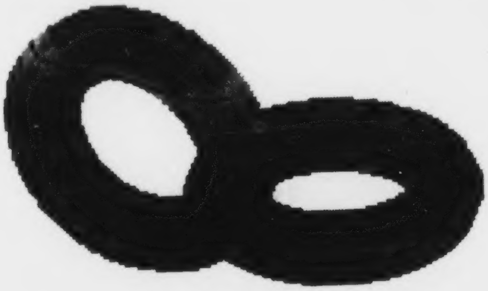
I see this task as a metaphor for the way in which the structure of attention shifts: how quickly and subtly, how obstinately, and so as a metaphor for how we come to know or perceive something differently in mathematics.

When I impose my ways of seeing I am stressing some features and ignoring others, but in order to survive, we have to do this for ourselves. So I want to invoke a second metaphor, namely framing.

I showed a reproduction of Emily Carr's painting Shoreline (1936). She was a West Coast painter (1871- 1945), contemporary of the Group of Seven. I then showed five potions of the paining, and the effect was as if to magnify that part, just as binoculars work in part by excluding, as well as by actually magnifying. By framing a portion of a painting you become aware of detail that may have been obscured in the more complex whole. This is what distinctions, principles, psychological positions, and propensities do for us: they reveal detail by restricting view.

Frames are metaphoric, for we all have frames through which we perceive the world. One of my principle aims today is to convince you that the notion of structure of attention can be a useful frame through which to look at teaching and learning mathematics, as well as itself being descriptive of how framing works.

The next exercise might give a taste of this. Imagine a torus, that is a solid inner-tube shape. Now imagine a second one identical to the first, and bring the two into touching contact. Now rotate one of them about the line perpendicular to the plane of their contact so that the two toruses are 'at right-angles' to one another. Now move one through the other until they intersect each other as fully as possible in one place.



Here is a picture of what I have in mind, but I am aware of the inadequacy of verbal description. Indeed I take every opportunity to try to describe without using pictures simply in order to work at my descriptive skills.

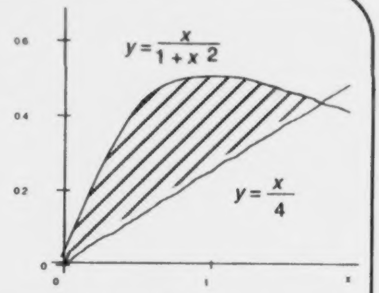
What is the volume of their common intersection?

I hope that question gave you a taste of what students experience when they are set tasks. There is often for them, as perhaps here for you, an initially impenetrable wholeness, a lumpishness. It seems hard to get any purchase, to find a way in.

Of course you immediately summon up connections. Perhaps you were reminded of the problem of the two (or even three) cylinders which is classic in early volumes-by-integration (calculus) and are hopeful that the strategy there might be helpful here. It might even be worth re-working it to see if the equations can be adjusted slightly. Or you might have thought about seeking a recognisable cross-section of the intersection.

The point is that there is a certain lumpish wholeness; details emerge, distinctions are made; properties (relationships) are recognised; activity begins; essence is sought.

Speaking of three cylinders, imagine three toruses intersecting mutually at right-angles. (It is worth trying to form an image before looking!)



Show that the shaded area is given by $\int_0^{\sqrt{3}} \frac{x}{1+x^2}\,dx - \frac{3}{8}$

This is an A-level exam question which I saw being used by a teacher. Only a few hours earlier I had suggested to the teacher the gambit 'Say what you see (and see what is said)' which I used earlier.

It emerged that most of the students were transfixed by the square root of 3, to the extent that many could not see anything else! What would you like them to be stressing? What would you like them to be focusing on? How does your attention change as you let your mathematicalness proffer foci?

Again there is lumpishness which for us is at most, fleeting. We distinguish components: graph, integral, equations, and without even being aware of it, have connected area and integral, area between curves and difference between integrals. The root 3 is of no concern. If we think about it at all it is just the x-coordinate of the point of intersection. The 3/8 is actually an area, perhaps an integral. And so on. Distinctions, relationships and properties, activity, justification, essence, all come as one package.

This is the structure of attention we want our students to experience.

In her study of adult students in remedial algebra classes, Mercedes McGowen (1998) found that students whose concept-maps grew in complexity over time (whose attention became more complexly structured) were also the ones who showed evidence of progress and who passed the course. Those whose concept-maps seemed very different each time they were asked (whose attention seemed not to grow in structure), were the students who did not succeed.

The inunction say what you see is an action which students can usefully internalise and make use of for themselves. It is useful as a means for focusing attention by first exposing multiple foci and then working at one or two of these. Not only does it focus attention, it also supports work on the importance of multiplicity of viewpoint, and on how to suppress your own interpretation while listening to someone else's.

SUMMARY SO FAR

I hope that you have had at least a taste of some of the aspects of the structure of your attention.

En bref
Summary So Far

Wholeness –Monadic– Totalité
Distinction, Tension –Dyadic– Distinction, Tension
Action, Relationship –Triadic– Action, Rapport
Activity –Tetradic– Activité
Essence –Pentadic– Essence
Present Moment –Hexadic– Moment présent

. . .

Where sourced? How sharp? D'où? Degré de précision?
What scope? What quality? Étendue? Qualité?

##

Lumpishness is perhaps more sensibly referred to as wholeness, as monadic, with connections to Leibniz in particular. By contrast with the immediate experience of lumpishness, of intractability, search and re-search seek a sense of wholeness arising from or as a result of having worked through making distinctions, locating properties (relationships within), participating in actions, engaging in activity, and so on.

You have, I think, found yourself making distinctions. When one feature or aspect is discerned, a distinction has been made. You may also have experienced tensions such as between wanting to think your own thoughts and having mine forced upon you, or being in lecture format but being asked to work as if in a workshop. Distinctions are the result of creative acts: they bring into existence. It is not so much a matter that 'I make a distinction', but rather that a distinction comes into existence as part of the structure of my attention. Maturana and Varela (1972, 1988) refer to this as autopoesis: a self making itself. Gattegno (1987) based his science of education on the challenging autopoetic assertion "I made my brain."

You have, I hope, been involved in several actions, though I haven't specifically pointed them out. This present experience and your past experience, mediated by your desire to construe what I am saying, may have produced an action inside you yielding a sense of insight or of sense-making. In the lecture we moved between modes of interaction: mostly I took the initiative, members of the audience responded, with the mathematics bringing us together. Sometimes the audience acted as mediating agent, bringing me in contact with some mathematics. I believe that for some people there was at times a welling up of a desire to express what they were seeing, which was suppressed by the lecture format. These are examples of six different modes of interaction between tutor, student, and content within a context (Sfard et al 1998).

You have, I hope, been engaged in activity, which like each of these kinds of attention, has its own internal structure, though there is not space to pursue details here, nor the more complex structurings.

You may recognise aspects of Anna Sierpinska's analysis of the concept of function (Sierpinska 1992), and you may detect resonances with the work of Pierre and Dina van Hiele in school geometry (van Hiele-Geldof 1957, Mayberry 1983, van Hiele 1986) and with David Tall's analysis (Gray and Tall 1994, Tall 1995). You may recognise a certain enactive resonance via Maturana's notion of poesis (Maturana and Varela 1972, 1988). The structures to which I have been pointing actually have very ancient pedigree

(Bennett 1966). But pedigree is less important than recognition in your experience. Can you become aware of these different structures of attention?

Also there is the question of locus, focus, quality, and scope. Sometimes your attention has been tightly focused or concentrated, as when trying to see the close-packed circles in different ways; other times your attention has drifted, and while hearing me speak you have thought other thoughts, following your own interests and connections.

And finally, when you become aware of your attention, you can try to observe whether you feel that it is centred or sourced inside you, behind you, or in front of you. You know the experience of someone who is very 'up front' and out-going. Their attention is often experienced as out in front of them, so they are constantly leaning forward metaphorically if not literally. Other people experience their attention as behind or at the back of their head. These people are reticent, often rehearsing internally what they say before saying it. Attention can also be more visceral, or more feelings oriented depending on circumstances.

I say this merely to point out that attention has a very complex structure.

So What?
Et alors?

Preparing to teach a topic
La préparation à l'enseignement d'un sujet
Constructing tasks & assignments
L'élaboration des tâches et des travaux
Listening to what students say
À l'écoute des élèves
Grading what students do
L'évaluation des travaux
Being sensitive to structure of students' attention by being aware of the structure of your own attention
Être sensible à la structure de l'attention de vos élèves en étant conscient de la structure de votre propre attention

So what?

I suggest that becoming aware of what you are attending to is fundamental to teaching any subject, and particularly so in mathematics. Unless the expert works at bringing to attention the shifts which have become second nature, even intuitive, it is difficult to appreciate what students are really saying and doing.

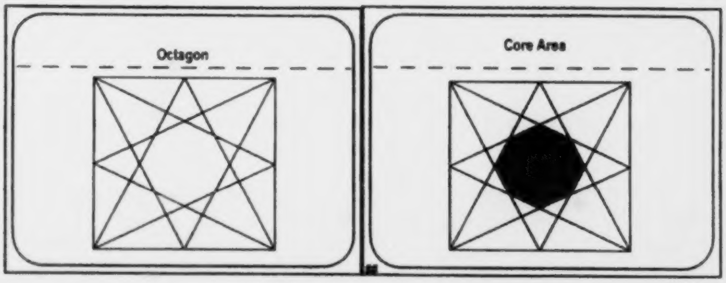
Furthermore, if you are unaware of how and where your own attention is structured, you are unlikely to be in a position to decide how and where it is most advantageous for students to place their attention. Thus, in order to become more sensitive to students, I find it helps to become more sensitive to myself.

CASE STUDIES: A PROBLEM AND A TOPIC

Let me now apply these ideas to a problem and to some topics. First a problem

Imagine a square, with the midpoints of the edges marked. Now join each midpoint to the two vertices opposite.

It is more effective to work at your own image before looking at my picture, if only to experience the strong pull of a provided diagram, and the different kind of work entailed in making sense of someone else's drawing.



Look at the central region formed by the inner lines. They outline an octagon.

Just for a moment your attention was single, narrowly focused, and sharp; the fact of my drawing attention to your attention diffuses and multiplies it, for you can be aware of the octagon but also aware that you are reading, that you are in a hurry to do something else, that your left foot is placed in a certain position, and so on.

Now see the octagon as equilateral.

Your attention locally moves about looking for relationships, for facts about segments in response to my suggestion (unless of course you already recognised the figure).

Your eye catches some right-angled triangles (or ones that look right-angled), that is distinctions are made and properties conjectured; you recognise a need to check that they are indeed right-angled (a shift to activity); or perhaps you become aware of the symmetry of the square, not as focal attention, but as background, within which you can see that certain pairs of adjacent edges of the octagon are equal, then that all adjacent pairs are, and so you see the octagon as necessarily equilateral.

The structure of your attention altered, in several ways.

- it became multiple alternating with single;
- it varied between broad and narrow;
- it varied between diffuse (sense of whole, of multiple relationships) and focused.

Sometimes you, as it were, wait until a thought, a relationship comes to you; other times you actively search for relationship, expecting that facts and relationships (similar triangles, symmetries, Pythagoras) will come to mind as required.

Sometimes you are aware of focusing on a distinction between two aspects or details, recognising two parts or a part and the whole;

Sometimes you are trying to relate two aspects, which requires awareness of the two parts and also of some aspect or feature that relates them. Mostly you expect this will happen for you. It is only when it stops happening that you get stuck, become aware of something being not quite right.

Perhaps you are already aware that this figure is a classic. It is an example of an equilateral octagon that is not equi-angular.

Now let's look again at the octagon. I think you will agree that it is not obvious that the square has 6 times the area of the octagon. What is interesting about the 6? Where is the infinity? Would you be surprised that the corresponding construction for the regular hexagon (how is your attention structured now?) is 14? Or that those are the only two I have found which give integer ratios? What happens when you change the mid-point to a general ratio?

What if you take a more general quadrilateral? We are interested in ratios of areas, so any affine transformation will serve. Parallelograms are in the affine equivalence class of squares, so the ratio of areas will be 6 for any parallelogram. But there are other quadrilaterals for which the ratio is also 6.

I want to report that I encountered the square : octagon area ratio of 6 : 1 in a workshop at the ATM Easter conference. When I moved to the hexagon, I was led to the question:

Which hexagons can be obtained from a regular hexagon by affine transformations?

But I can report that it took me quite a while (several sessions of thinking about it) to shift from holistic conjectures to relational, analytic thought, in order to characterise those hexagons. I kept thinking that a certain property would be enough (opposite edges equal; opposite edges equal and parallel, diagonals intersect in a single point, diagonals parallel to pairs of edges, ...), rather than writing down all the affine properties of a regular hexagon I could think of and then seeking a minimal generative set. Wanting to depict all such hexagons using Cabri Géomètre oriented me towards a minimal generative set, but the presence and ease of use of the software enticed me to try conjectures rather than to be comprehensive and analytic.

Now let me slip into metacognitive mode. Where did my questions to you come from? I suggest that they are the effect of a pedagogic action following an awareness of a shift of attention. What I mean is, my attention became absorbed by the 6, and by not even hoping to see why it is an integer. Attention on integers raises the question of whether this construction is special. In other words I became aware of other possibilities 'like' this one, and these I posed to you as questions.

So I propose the conjecture that questions and probes and prompts to students arise when I experience a shift in the structure of my attention of what I am stressing, and this is transformed by my pedagogic self into a question. (See also Watson and Mason 1998)

What has all this to do with teaching? I hope you can see that it has everything to do with teaching; that without being aware of the structure of your own attention, you are unlikely to be sensitive to what your students may be stressing and ignoring. If you are not working at getting your students to stress and ignore, to structure their attention appropriately, then there is a good chance many of them will not ælearnÆ from your intervention.

Let me take a generic meta-example:

Exemples
Examples

What are examples examples of?
Qu'exemplifient les exemples?

##

I have been offering you tasks to work on, and then telling you things that I think are related to and which generalise what you experienced. I was offering you examples. But you are faced moment by moment with either being fully involved within 'the activity' (eating of the sweet fruit), or deciding what it is that is supposed to be being exemplified.

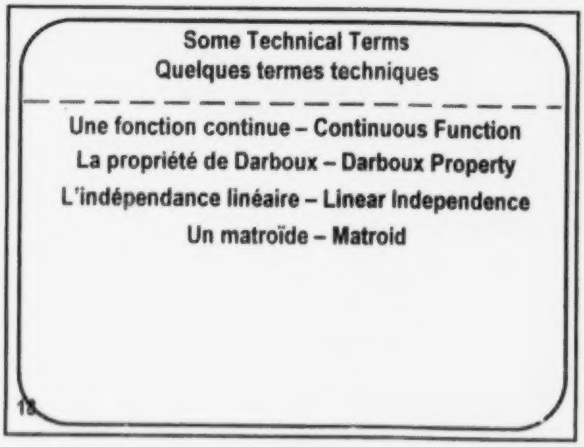
I am sure you have found that difficult, perhaps even have been mystified. Certainly students experience that!

Turning a particular, whether it is a mathematical object, a task, or a worked solution, into an example of a class of objects, a type of task, or a technique, requires a shift of attention. It requires that the student stand back from 'eating the sweet fruit' and perceives generality through the particularity. Most students require specific direction to do this, at least for a while until it becomes established as their own practice, and freshly when working in a new context. Riemann and Schult (1996) provide a survey of research into students' use of analogical reasoning within a domain to help them to interpret a contextualised problem in mathematical terms, to decide which technique to apply when several are possible, and to generalise (p124).

Sandy Dawson showed me some notes written by a student near the end of her course in which she observes that throughout the course Sandy used children's stories to introduce mathematical topics, yet she has only just recently recognised this as a teaching technique. Like the proverbial fish ("if you want to know about water, don't ask a fish"), she has been immersed in a practice but unaware of it as a practice. I submit that this happens for many students in mathematics at every level.

Suppose I work a question publicly in front of my students, or write out a good resolution for them. What do students see as exemplary about my example? Can they see it as an example if they do not yet appreciate the generality of which it is the particular?

Let me turn now to a second form of example of shifts of attention, namely technical terms. Here are four:



Some Technical Terms
Quelques termes techniques

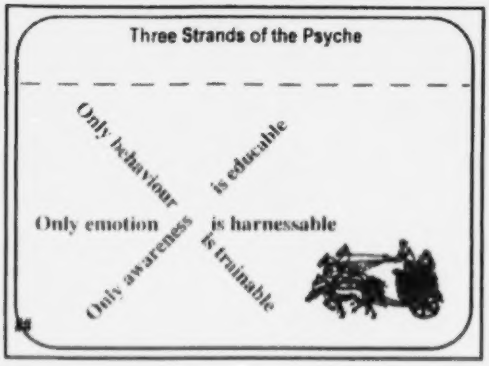
Une fonction continue – Continuous Function
La propriété de Darboux – Darboux Property
L'indépendance linéaire – Linear Independence
Un matroïde – Matroid

What do you find yourself doing when you read them?

I suspect that for familiar ones you experience at most a sense of slightly more focused potential. You know a great deal about them and you wait to see what aspects I might call up. You may catch a sense of details and connections coming closer to the surface waiting to be activated by what comes next.

For others that are less familiar you may go so far as to experience a hiatus, a gap, even a tightening of muscles in the gut!

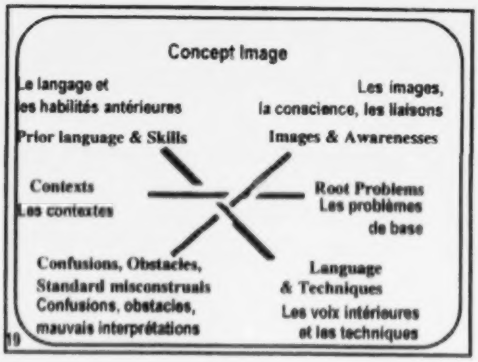
As Tall and Vinner (1981) pointed out, we each have a concept-image consisting of a collection of words, images, concepts, theorems, problems, contexts, etc. which McGowen (1998) nicely describes as a collage. My colleagues and I have found it convenient to locate or impose structure on concept-images, which reflect something of the structure of the psyche and hence the structure of attention. The result is a framework that informs my preparation for teaching a topic or appreciating a topic (Griffin and Gates 1988).


Three Strands of the Psyche

Only behaviour is trainable

Only emotion is harnessable

Only awareness is educable


Concept Image

Le langage et
les habilités antérieures
Prior language & Skills

Les images,
la conscience, les liaisons
Images & Awarenesses

Contexts
Les contextes

Root Problems
Les problèmes
de base

Confusions, Obstacles,
Standard misconstruals
Confusions, obstacles,
mauvais interprétations

Language
& Techniques
Les voix intérieures
et les techniques

The three interwoven axes correspond to the three traditional aspects of the psyche, here rephrased in terms which are closer to the Upanishadic sources than the currently popular terms enaction, affect, and cognition. I have also expressed them in a form drawn from Gattegno's memorable assertion that "only awareness is educable" (Gattegno 1987).

In behavioural terms, each topic, each technical term draws upon behaviour already trained, represented both as actions and as language used with some facility though perhaps with informal or non-technical meanings. For example, before students encounter linear independence, they are familiar with the notions of independence and dependence, even if in rather imprecise form.

Each topic and each technical term are also associated with techniques, and these techniques have 'inner incantations' which are idiosyncratic for the most part, but which help direct attention while the technique is being performed. There are of course technical terms associated with the techniques and the topic. In the case of linear independence these include testing for linear independence, the notion of a basis, and so on. In testing for linear independence there is a potential incantation which is a rehearsal of the meaning of independence: "if these vectors are linearly independent then any linear combination must in fact be the all-0-combination, so I form the equations and ..."

In the motivational-affective dimension, each technical term arises because it is needed to point or focus attention in some context in response to some root or source problem or question. Thus it makes sense to renew contact with those root problems, and even if they cannot be stated in a form accessible to students, some version probably can, otherwise it may not be appropriate to be teaching this topic! Furthermore, students can be given some indication of the variety of contexts in which the term or the topic is likely to arise.

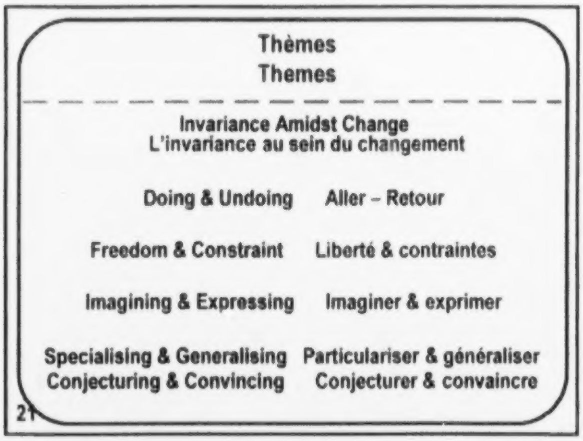
In the awareness-cognitive dimension, there are images, (visual, visceral, and 'senses-of'; diagrammatic, spatially metaphoric, sensory, symbolic, etc.) and connections to other terms which are triggered metonymically and resonated metaphorically when the topic or the term comes to mind, but which aspects are dominant depends largely on context and 'frame of mind'. There are also standard construals which do not fit with mathematical usage (misconstruals, obstacles, misconceptions), although often their construction exhibits excellent mathematical thinking applied to inappropriate data. For example, 0.3 x 0.3 is 0.9, reached from the sequence 0.5 x 0.5 is "oh point twenty-five"; 0.4 x 0.4 is "oh point sixteen", ...). One of the important ways in which the human brain seems to work is through the use of labels, which act not just as summaries but as axes around which experiences can be collected and triggered (Mason in press). One instrument often employed to investigate these collections and webs is concept maps (Novak 1990, Spence 1996, McGowen 1998) or topic webs (Buzan 1993, Buzan and Buzan 1994).

The threefold framework is not intended as a template or a rigid scheme. Used mechanically it will be as useless as any other straitjacket. The purpose of a frame like this is to sink into the background and to emerge into consciousness only when you feel something is missing but are not quite sure what. Over time I have integrated this framework into my functioning, so that sometimes when I feel I might be *missing something I recall the framework to mind and interrogate it in my particular context, but mostly* it in-forms without being a pro-forma. It is maximally effective when it presents in some manner the complexity of your awareness of your awareness.

This is an example of a complex frame, in the sense we met earlier. It is how I frame the question of considering a task or structuring a presentation. Being aware of some of the frameworks that structure and inform actions, the question arises as to how it is that decisions are made, whether during preparation or in the midst of an event. I suggest that they are mostly made hours, weeks, years before when certain habits are established. And as the Zen saying has it,

Habit forming can be habit forming.

When decisions are made in the moment (whether during preparation or during the flow of an event, or in retrospect) they are made on the basis of what we are aware of, not necessarily consciously.



Thèmes
Themes

Invariance Amidst Change
L'invariance au sein du changement

Doing & Undoing	Aller – Retour

Freedom & Constraint	Liberté & contraintes

Imagining & Expressing	Imaginer & exprimer

Specialising & Generalising	Particulariser & généraliser
Conjecturing & Convincing	Conjecturer & convaincre

21

For example, the degree to which we are attuned and sensitised to mathematical themes such as those listed in the slide (as well as heuristics, thinking processes, techniques, etc.) will influence what we 'think' of pursuing in the moment. If a particular theme comes to mind it might suggest pausing and being explicit with students, or it might suggest an exercise or task which might bring that theme to students' attention.

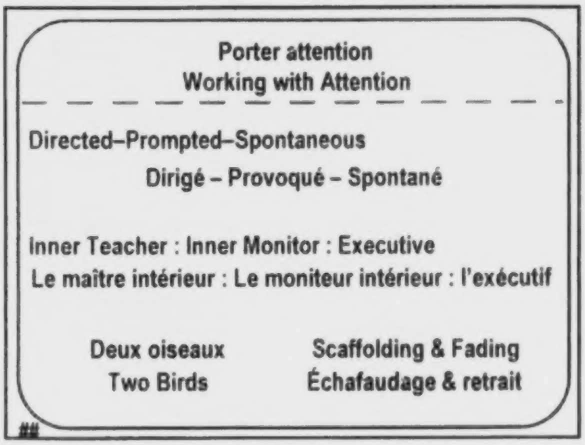
But if a theme is not part of your attention, you are unlikely to have the opportunity to choose to work on it explicitly or implicitly. Some students may become aware of such a theme spontaneously through their own construal, but the majority of students are most unlikely to if it is not part of your attention.

WORKING WITH ATTENTION

As David Wheeler pointed out in his Topic Group, there seems to be a paucity of attention devoted to the technical details of teaching acts, that is, of what a teacher does with themself, with their attention, so that student powers are invoked, student energies are evoked, and student attention is provoked (into being structured in certain ways). In his Ad Hoc session, Dave Hewitt demonstrated a few of the ways in which a teacher can arrange that student attention is placed usefully without using explicit words. His criterion for a teacher technique is that it cannot be written down in the student text as something for the student to do. One of my long term projects, which I call Meaning Enquiry in Mathematics Education, or MEME's (as in Richard Dawkin's sense of a cognitive analog to a gene) (Dawkins 1976), is to collect teacher techniques, gambits, stratagems, frames, frameworks, etc. (memes) and to make them available.

The interesting part is working at ways of making a meme available to someone else, and this draws upon the Discipline of Noticing (Mason 1996).

My colleagues and I at the Centre for Mathematics Education at the Open University have developed ways of offering frameworks to colleagues at a distance, based on undertaking tasks to generate immediate experience for oneself, then relating that experience to some classroom context. The framework acts as a focus around which a variety of recent and past experiences can accumulate, ready to be triggered both individually and as a whole (Mason 1989). One of the frameworks, actually a meta-frame, which we have offered teachers (EM235 1984) is a version of scaffolding and fading (Wood et al. 1976, Collins et al. 1989, Love and Mason 1994), in which the teacher is at first fairly explicit and directive about the use of a particular term, a particular thinking process, technique, etc., then begins to become less and less direct in their prompts as students increasingly spontaneously use it themselves. The teacher is acting in the way Bruner (1986 p75) describes as being "consciousness for two", doing for the students what they are not quite yet able to do for themselves (attend to processes or to global goals while engaged in particular tasks). When the students talk over the use of a particular process, term, etc., the teacher can attend to something else, working through the same sequence with the new term, process, heuristic, etc. For example, by using a format of questions consistently, then drawing attention to that form while fading the support, students may come to experience mathematical thinking and to internalise those questions.



Porter attention
Working with Attention

Directed–Prompted–Spontaneous
Dirigé – Provoqué – Spontané

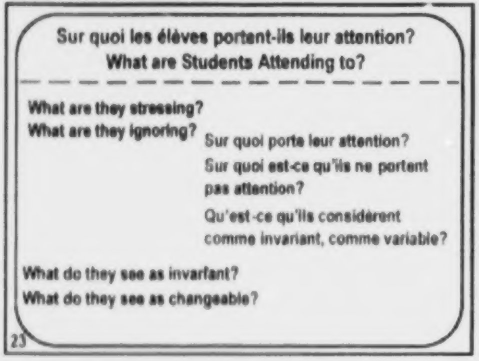
Inner Teacher : Inner Monitor : Executive
Le maître intérieur : Le moniteur intérieur : l'exécutif

Deux oiseaux
Two Birds

Scaffolding & Fading
Échafaudage & retrait

The frame Directed—Prompted—Spontaneous acts as a reminder to fade, for scaffolding can only be said to be accomplished when fading has been completed and the students' awareness educated. Another way of thinking about the same process is as awakening the second bird, building an inner teacher (Hyabashi and Shigematsu 1988), inner monitor (Mason et al. 1982), or inner executive (Schoenfeld 1985).

Returning then to my opening and abiding questions, I hope I have convinced you that the question of what students are attending to, what they are stressing and what consequently they are ignoring, what features they see as invariant and what is permitted to change, is of major importance. Furthermore, in order to address these questions and in order to act in such a way that they are likely to be attending to what you are attending to, it is useful to become aware of what you are attending to, of the structure and form of your own attention.

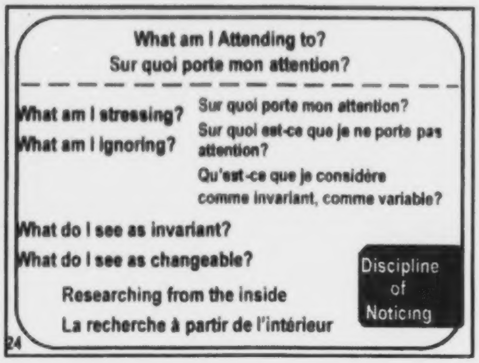


Sur quoi les élèves portent-ils leur attention?
What are Students Attending to?

What are they stressing?
What are they ignoring?

Sur quoi porte leur attention?
Sur quoi est-ce qu'ils ne portent pas attention?
Qu'est-ce qu'ils considèrent comme invariant, comme variable?

What do they see as invariant?
What do they see as changeable?



What am I Attending to?
Sur quoi porte mon attention?

What am I stressing?
What am I ignoring?

Sur quoi porte mon attention?
Sur quoi est-ce que je ne porte pas attention?
Qu'est-ce que je considère comme invariant, comme variable?

What do I see as invariant?
What do I see as changeable?

Researching from the inside
La recherche à partir de l'intérieur

Discipline
of
Noticing

Teaching consists of acts undertaken by a teacher, but whether learning results is another matter. I find myself constantly brought back to the question of what students are attending to,

when I am expounding,
when they are exploring or doing exercises,
when I am explaining something,
when they are responding to assessment.

Addressing these questions is not an empirical matter in the sense of surveys and interviews with others. The most reliable methods I have found start by turning the questions back towards myself.

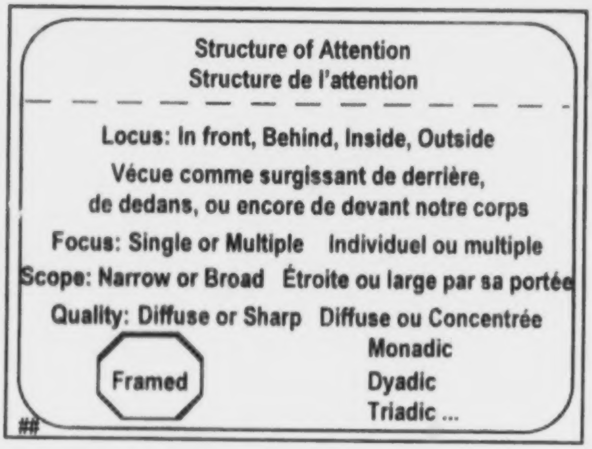
The methods I have garnered from many sources constitute what I call the Discipline of Noticing, which is one way to Research From The Inside, that is, to become aware of my own experiences before trying to recognise or observe others. I do not intend to inflict any theory about this upon you, which would take more than a lecture in itself. But I hope I have illustrated some aspects through the way I have invited you to work this afternoon.

Gattegno (1987) offered an instance when he wrote:

> Since on the whole we are not required to study learning while learning, we can propose to teachers that they deliberately become students for the sake of knowing what learning is, first hand, provided that, they do not get lost in the acts of learning, but watch them instead. (p164)

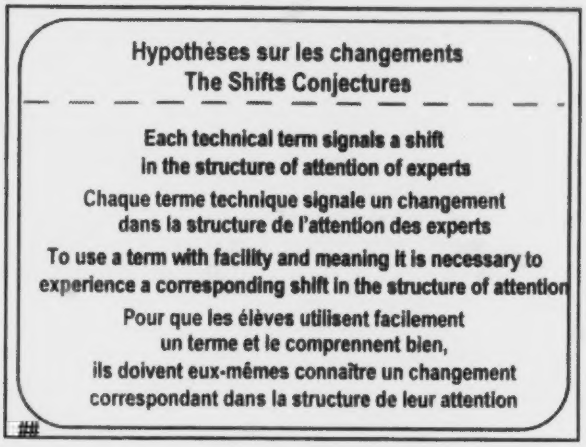
In our working group Olive Chapman gave a very nice example of what I mean by researching from the inside. Before sending master's students out to interview children and teachers, she interviewed those students herself. They could then experience being interviewed while they were preparing to interview, in order to develop sensitivities to their interviewees. In the same way, I find that to address any question in mathematics education it is vital to locate some parallel personal experience, preferably recent. Thus if I want to work with students on introducing symbols, it helps me to work on a task which involves me introducing symbols but where for some reason I need to struggle a bit (Mason 1989).

SUMMARY


Structure of Attention
Structure de l'attention
Locus: In front, Behind, Inside, Outside
Vécue comme surgissant de derrière, de dedans, ou encore de devant notre corps
Focus: Single or Multiple
Individuel ou multiple
Scope: Narrow or Broad
Étroite ou large par sa portée
Quality: Diffuse or Sharp
Diffuse ou Concentrée
Framed
Monadic
Dyadic
Triadic ...
##

In summary then, attention is complex. It has locus, focus, scope and quality, and it has structure (monadic, dyadic, triadic, etc.). It frames or is framed by frameworks, theories, beliefs, perspectives and world-views which have been internalised.

The exercises and observations so far lead me to two conjectures concerning shifts in the structure of attention:



Hypothèses sur les changements
The Shifts Conjectures

Each technical term signals a shift
in the structure of attention of experts

Chaque terme technique signale un changement
dans la structure de l'attention des experts

To use a term with facility and meaning it is necessary to
experience a corresponding shift in the structure of attentio

Pour que les élèves utilisent facilement
un terme et le comprennent bien,
ils doivent eux-mêmes connaître un changement
correspondant dans la structure de leur attention

Davis (1984) notes the cognitive shift in which a verb becomes a noun, that is, a process becomes an object, and many authors (Sfard, Dubinsky, Tall, Sierpinska, among others) have elaborated on ways in which processes become objects and abstractions become reified into objects. Sfard (1992) distinguished *interiorisation* (in which processes are performed on familiar objects), *condensation* (a term taken from Freudenthal, in which a process is compressed into an entity so enabling it to be thought of as a unity rather than as a sequence of actions), and *reification* (in which a condensed process becomes an object of study in its own right). In Floyd et al. (1981) we developed a spiral image with an accompanying three-term framework for use by teachers in order to remind them in the midst of their teaching or their preparation that 'Things take time', and in particular, people naturally turn to confidently manipulable entities, whether apparatus, images, or symbols, so that while manipulating them they can get a sense of what is going on, and after some trials, begin to bring that sense-of to articulation. Articulations comprise or refer to more entity-like conceptions which in turn may become confidence inspiring and confidently manipulable entities. Thus emerged a frame or framework: *Manipulating—Getting-a-sense-of—Articulating.*

Dubinsky (1991) also proposed a three-stage theory in which actions are at first holistic, but as attention is freed due to familiarity, distinctions of stages or steps emerge from which comes *processes,* which with facility gained through use and further freeing of attention, become objects of study.

Tall (1995) suggested that rather than a cyclic developmental process, conceptualisation comes about through a mutual co-emergence of process and concept, hence the combined term *procept.*

The shifts conjectured are not proved mathematically, nor are the proved through quantitative empirical studies. They are validated not globally for all people at all times in all places and under all conditions, but locally for you in your situation now. They are validated in and through your experience, according to whether you find that something sticks with you and begins to in-form your practice, to educate your awareness. For in order to be able to make conjectures about what students are attending to, it is vital to be aware of where your attention is placed. Then you can address the question of where you want your students' attention to be, and which techniques will make it a strong possibility that that is where their attention truly is.

THE ROLE OF WILL

I have deliberately not offered definitions, because of my belief that terms such as attention and awareness are understood through use related to experience, not through formal definitions. After the lecture I was challenged to distinguish attention from awareness, and my considered reply was that attention is the manifestation of will: we are present in and through our attention (we literally attend), we are where our attention is, and the shape or structure of our experience is the structure of our attention. Awareness on the other hand, while having a dictionary definition of *consciousness,* applies more generally to what lies below the surface of attention, what is poised or buried that could come to attention or could influence the structure of attention. Thus to a child with a hammer, the world consists of things to hit: the presence of the hammer in the hand activates awarenesses which are manifested as muscle control producing hitting. Thus we need not be present in our awarenesses, which may be the actual or potential placing of attention, which then becomes our presence.

The connection of attention with will has roots in ancient Indian writings. David Wheeler drew my attention to a modern source, Samuel Coleridge, (1983) as saying:

> But the will itself by confining and intensifying the attention may arbitrarily give vividness or distinction to any object whatsoever.

To become aware of will itself is rather difficult, given that it manifests itself as attention, though Gattegno (1987) offers an exercise through which to identify will:

> I can become aware that my eyes move or, more precisely, that I can act on some of my ocular muscles to move my eyes ... and that that movement can be made very gradual and slow, and through generating such actions I can become aware of my will as it commands my eyeballs to move. Therefore I am not only with my eyes, I am with my will as well. (p38-39)

Habits can be seen as evidence of will, of the individual having made choices in the past. The notion of structure of attention, and the shifts conjectures, are based on the same perspective: each individual makes choices, however constrained by circumstances, and it is in and through those choices that individual freedom is to be found. Teaching mathematics is a domain in which it is possible to experience the making of fresh choices in the moment, triggered in part by circumstances, in part by sensitivities developed through intentional noticing and reflection, in part by interest in what students are attending to and with what structures, and in part by awareness of the structure of one's own attention.

ACKNOWLEDGEMENTS

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Plenary Lecture 2

COMMUNICATING MATHEMATICS OR MATHEMATICS STORYTELLING

Katherine Heinrich,
Simon Fraser University, BC

The greatest compliment I ever received from a student was "I can't believe it. I look forward to coming to class." This was a student in a university mathematics course for those intending to be elementary teachers, who believed she had between little and no mathematical interest, knowledge, ability or confidence. She was amazed that she (or anyone like her) might find some pleasure in mathematics—might enjoy the class. We are all aware of the poor reputation of mathematics, that it has frustrated, belittled and distressed far more than it has brought joy and wonder. While we accept that not everyone will want to study advanced mathematics, we must not accept that they will not enjoy and be entertained by some aspects of mathematics (few of us enjoy all mathematics). Perhaps if we made a deeper commitment to telling our stories we could change the negative views many have of mathematics. We have all enjoyed stories from disciplines in which we have no expertise: economics, biology, history, psychology, art and music. Why shouldn't others have an opportunity to enjoy the stories of mathematics?

Unfortunately, it can be a very difficult story to tell well. Our words, our notation and our language are significant challenges to any storytelling. We have built a language and system of notation that allows us to express ideas succinctly and precisely. In 1782 Euler gave a 100 page proof of the existence of certain families of orthogonal latin squares. Using modern notation, and the current expectation that we be as concise as possible, this same proof can now be presented in about a page. This works well for those who understand the language, but is completely inaccessible to everyone else. To tell our stories we need to speak in the
language of our audiences - and as our audience changes so too must our choice of words.

To illustrate. Probably the most difficult talk I ever gave was a twenty minute talk to an elementary school. As I sat down to prepare the overhead slides, I suddenly realised my audience couldn't read—the entire talk would have to be given in pictures and speech. And so it was and I was pleased that I had appreciated the nature of my audience. Well, almost. We were looking at Escher tilings and to show them how, in one tiling, each fish exactly made up a square, I picked up one of the fish and cut it into pieces only to hear alarmed screams of "She's cutting the fish!" But a year later one little girl still remembered the talk.

It is extremely difficult to recognise when we are using words from the vocabulary of mathematics. A few years ago I assisted in making a postcard for a mathematics and art presentation. Doris Schattschneider was to tell her wonderful tale of Escher, Marjorie Rice, Art and Mathematics—a talk accessible to the broadest of audiences. I wrote "... tile the plane." The designer looked at me and said "A plane. What do you mean, you're tiling a plane?" For her a plane has wings and takes you places.

One could make definitions, but an audience unfamiliar with them will quickly forget and lose interest. We are fortunate in mathematics to be able to illustrate our presentations with pictures and draw upon analogies. Our audience has an excellent ability to imagine, to abstract and to generalise from specific examples—far more than they usually realise and than we usually give them credit for. As long as we provide clear guidance and a good, interesting story-line they will follow us.

Yet it remains challenging and we must add to the challenge of actually telling the story, the challenge of a culture that for many years has had a negative impact on the telling of our stories, both to others (the so-called "general public") and to one another. Over the years we have built an environment and culture in which it has become acceptable to give presentations at conferences that few of the audience have any hope of understanding. For a long time this culture has implied:

- Once invited to give a talk it is important to justify the invitation by showing how much you have accomplished.
- If a presentation is clear and comprehensible to everyone, it can't have had any content.
- There are areas of mathematics so deep it's not possible for anyone without extensive knowledge in the area to know them.
- People who spend their time on public talks and promoting mathematics do so because they can't do "real" mathematics.
- If we don't show all the details and the audience doesn't learn some mathematics it isn't really a mathematics talk.

Our goal must be to rewrite these statements. Perhaps as: * If you have been invited I know it is because you have done great things. Now I want an insight into your area of mathematics—why it is interesting, why it excites you, how it fits in. I don't need details.

- If a presentation is clear, interesting and understandable to everyone in the audience, it is an outstanding presentation.
- There is no area in which, with hard work and careful thought, some part cannot be made accessible.
- People who spend their time on public talks and promoting mathematics are skilled people who do the discipline a great service.
- It is possible to appreciate mathematics and see its beauty and utility without having to learn it.

Again, to illustrate. Many years ago my husband and I gave talks at the Hungarian Academy of Sciences. We were both extremely nervous about speaking to such an esteemed group of mathematicians, but went ahead and gave our talks. We have both for a long time tried to give talks accessible to the non-specialist and were afraid this audience would think it all so simple and be bored. At the end of our talks a member of the academy came to us and said "We really enjoyed your talks. So often our visitors think we are brilliant and so feel they must give very high level technical talks. We never understand them." Not so long ago, a new PhD preparing a presentation on the work in his thesis looked at me after we had gone through a very good practice and said, "But it's easy. No-one will think I did anything." How easy it is to forget what depth of understanding comes after four years of thinking and how hard to recall the difficulties in learning new ideas.

These are the challenges: the slowly changing culture and the commitment of time and energy to give successful mathematics presentations. We must respect and support our colleagues who take on such tasks. Continuing success in the latter will indeed change the culture—we may even all discover the joy of telling our mathematical stories to appreciative audiences.

BEST COPY AVAILABLE

Now appears to be a particularly good time to become a proficient mathematics storyteller. Of late I have read and heard repeatedly about a new general interest in mathematics. There was for example the PBS series "Life by Numbers," the movie "Goodwill Hunting", and the one I heard of most from friends and colleagues, the Nova documentary "The Proof" which tells the story of Andrew Wiles and his proof of Fermat's Last Theorem. For many, their interest was not the mathematics, but the enormity of Wiles' accomplishment, his dedication and his passion - in other words, his story.

And there are more glimmerings of hope. After a recent Canadian Mathematics Society prize lecture one of my colleagues said "The speaker broke all the rules. He tried to give a presentation we could understand." Organisations such as the CMS and the research institutes are not only starting to talk about the importance of good mathematical storytelling, but are incorporating public presentations into their conferences, organizing events aimed solely at "popularizing" mathematics, and searching for ways to ensure more people have an opportunity to enjoy and appreciate mathematics. But all such initiatives require the time and dedication of many, many mathematicians—and the support of all mathematicians.

I'd like to now give you some examples from talks that have stayed with me over the years; not talks that I necessarily gave. The first was at an international discrete mathematics conference. The speaker proved a theorem; usually a sure-fire way to destroy the audience, but in this case we saw a result of extreme clarity and beauty.

Suppose you have a whole bunch of points and every pair of them is joined by a coloured line; the colour being either red or blue. It is true (as we shall see) that no matter how the lines are coloured it is always possible to separate the points into two groups so that in each it is possible to travel from one point to the next, to the next and so on until all points in the group have been reached using only a sequence of red lines and in the other group, the same can be done with blue lines. These sequences of red lines and of blue lines we will call paths. (A small technical point that must be noted, but which I will not again refer to, is that one of the groups of points might have only one point in it.)

For example, shown below is one colouring of all the lines when there are six points (Figure 1) and beside it two paths that satisfy our requirements (Figure 2). In fact, there are many choices for the paths.

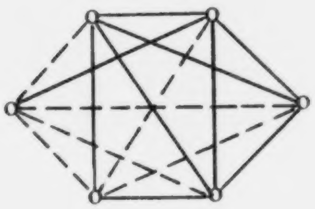


Figure 1

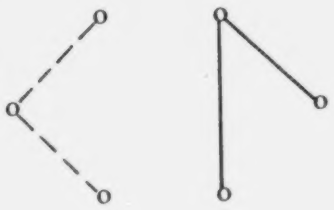


Figure 2

I hope the problem is clear. I am going to attempt to describe the proof in words as one might speak (but of course one would speak differently on observing how the audience was reacting). As you will see, with written words and fixed pictures, things quickly become quite cumbersome, but even still I hope the flavour of what is happening remains.

Suppose you choose one of the points and the colour red. Starting from that point make the longest sequence of red lines that you can (that is, the longest red path). At some point you will get stuck and not be able to add another red line. If that happens, then choose a point not on the path you have just made and in the same way, but using blue lines, join it to as many new points as you can (a blue path). At some point you will get stuck with this one too. What we now have looks like the situation in Figure 3 with a red path and a blue path.

Let us call the last points on each path a and b, respectively. If all the points we started with are in these two paths then we have accomplished what we set out to do. But suppose some points are left over. Choose one of these leftover points and call it x. We know the line from a to x is blue and not red as otherwise x would have already been added to the red path. Similarly the line from b to x is not blue but red. So we have Figure 4.

Now, what is the colour of the line from a to b? If it is blue, then we take the two new paths shown in Figure 5 and if it is red we take the two new paths shown in Figure 6. We repeat this process until no points are left over and we have the two paths we were searching for.

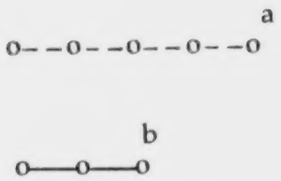


Figure 3

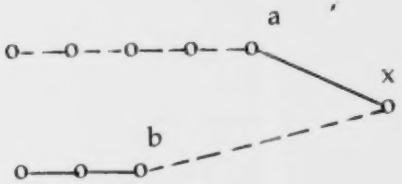


Figure 4

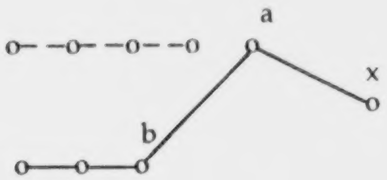


Figure 5

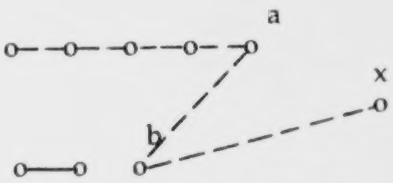


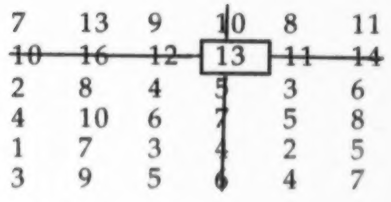
Figure 6

In the talk several more examples were presented, all with multicoloured pictures, the ability to move things around on the screen, and talking with and listening to the audience. I will not attempt to present them here. However, I will end with an example probably well-known to many of you. I have used this example with elementary students and with prospective elementary teachers. The teachers were intrigued by it. The students quite indifferent—I guess there is a time in one's life when "magic" is the norm.

Consider the array in Figure 7. Choose one of the numbers. Now delete all other numbers in the same row and column as the number you chose. To illustrate, in Figure 8 I have chosen 13 in the second row and fourth column and then deleted all other numbers in this row and this column. From what remains choose another number and delete all other numbers in its row and column. In Figure 9 I have chosen 10 from row 4 and column 2 and then deleted the other entries in this row and column. Continue choosing until you have six numbers. My choices were 13, 10, 3, 5, 7, 5 as shown in Figure 10.

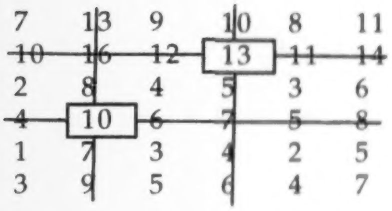
7	13	9	10	8	11
10	16	12	13	11	14
2	8	4	5	3	6
4	10	6	7	5	8
1	7	3	4	2	5
3	9	5	6	4	7

Figure 7



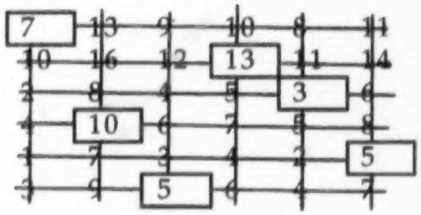
7 13 9 10 8 11
~~10 16 12~~ 13 ~~11 14~~
2 8 4 5 3 6
4 10 6 7 5 8
1 7 3 4 2 5
3 9 5 6 4 7

Figure 8



7 13 9 10 8 11
~~10 16 12~~ 13 ~~11 14~~
2 8 4 5 3 6
~~4~~ 10 ~~6 7 5 8~~
1 7 3 4 2 5
3 9 5 6 4 7

Figure 9



7 ~~13 9 10 8 11~~
~~10 16 12~~ 13 ~~11 14~~
~~2 8 4 5~~ 3 ~~6~~
~~4~~ 10 ~~6 7 5 8~~
~~1 7 3 4 2~~ 5
~~3 9~~ 5 ~~6 4 7~~

Figure 10

Add the numbers: 13 + 10 + 3 + 5 + 7 + 5 = 43. Amazingly the answer is always 43 no matter which numbers you choose provided you follow the directions precisely. Why? Rather than provide an explanation, let me leave you with Figure 11.

+	1	7	3	4	2	5
6	7	13	9	10	8	11
9	10	16	12	13	11	14
1	2	8	4	5	3	6
3	4	10	6	7	5	8
0	1	7	3	4	2	5
2	3	9	5	6	4	7

Figure 11

There are hundreds of examples like this through which one can talk about mathematics and elicit wonder, interest and enjoyment. In addition, there are the applications, histories and personalities of mathematics which can be discussed and described in stories both entertaining and accessible. There is no lack of material about which to talk and excite a variety of audiences. What has been lacking is a culture that values and supports such endeavours, and the people willing to take the time and energy to tell our stories.

WORKING GROUPS

Working Group A

ASSESSING MATHEMATICAL THINKING

Florence Glanfield, University of Saskatchewan
Pat Rogers, York University

PARTICIPANTS:

Jon Carrodus, Balmoral Junior Secondary School
Lydia De Santos, Kwantlen University College
Malgorzata Dubiel, Simon Fraser University
Don Eastman, Brandon University
Gary Flewelling, Arthur, Ontario
Doug Franks, Nipissing University
Sandra Frid, University of New England, Australia
George Gadanidis, Durham District School Board, Ontario
Pam Hagen, Coquitlam School District, BC
Kathy Heinrich, Simon Fraser University
Lars Jansson, University of Manitoba
Heather Kelleher, University of British Columbia
Doug McDougall, OISE/University of Toronto
Bruce McAskill, Ministry of Education, BC
Jim Mennie, University of British Columbia
Yvonne Pothier, Mount St.Vincent University
Tom O'Shea, Simon Fraser University
Graham Rankin, Kwantlen University College
Christine Shiu, Open University, UK
Shannon Sookochoff, University of Alberta
Susan Stuart, Nipissing University

INTRODUCTION

The purpose of this working group was to allow participants to work through their own ideas on the role of assessment in developing mathematical thinkers. Initially, to provide a focus for discussion, six questions were posed:

- When we speak of assessing mathematical thinking, what is it we are trying to assess?
- What is the role of assessment in developing mathematical thinking?
- What are some assessment techniques that group members have used, heard about, or want to know more about?
- What assessment techniques work best for gathering particular types of data about mathematical thinking?
- Once we have collected the data about mathematical thinking, then what?
- Should we develop guidelines around assessment of mathematical thinking?

As our conversation about assessment developed over the three days we constantly returned to these questions and refined them.

The term *assessment* is often used as though it were synonymous with *evaluation*. How do we use the term? NCTM defines assessment as the process of gathering evidence about a student's knowledge of, ability to use, and disposition towards mathematics and of making inferences from that evidence for a variety of purposes (1995, p. 87).

Gardner (1992) defines *assessment* as "the obtaining of information about the skills and potentials of individuals, with dual goals of providing useful feedback to the individuals and helpful data to the surrounding community."

The focus for the group was on assessment as a process that teachers use to collect information about students' mathematical thinking.

DEFINING OUR FOCUS

The first day was devoted to defining our focus. Preparatory work in small groups centred on two questions: What do I know about assessing mathematical thinking? and What am I wondering about? The following themes emerged:

- What is mathematical thinking? What does it look like?
- What questions might we design to give students opportunities to show mathematical thinking?
- Do teachers assess mathematical thinking or something else?
- What should we assess? A partial answer:
  - more than knowing the rules and algorithms, but when to use them
  - ability to communicate
  - how the use of calculators affects understanding of concepts.
- What shouldn't we do? A partial answer: not separate communication from math thinking/knowledge.
- Can you assess mathematical thinking if you don't have a good grasp of math concepts?
- Assessing differently in large classes in the context of obsession with grades.
- Continuum of mathematical thinking/continuum of different assessment techniques dependent on students' need to know the mathematics.
- Teachers decide what they are going to assess: they evaluate what students show them and we observe based on what they ask.
- Purposes of assessment: to make judgments; to make decisions about instruction; for program correlation.
- What makes us interpret content in the way we do? Should curriculum drive assessment or assessment drive curriculum?

- Knowing when you know: can we use self assessment to remove the need for teachers to assess?

- Knowing when you don't know: students' attitude in confronting this, confidence, and how to assess this.

These themes were the topic of the discussion for the next two days.

EXPLORING ASSESSMENT

The second day of the group was devoted to an activity that focused our attention on the use of criteria or scoring rubrics in assessing student performance. The discussion which ensued was very rich and is reproduced below with minimal editing and arrangement into thematic issues.

The Assessment Activity

Setting the Stage

When participants entered the room, they found it arranged with 4 chairs placed at the front of the room and the remaining chairs placed in a large semi-circle. The room had a decidedly theatrical appearance. Four volunteers were called for to act as judges and four volunteers to act as performers. The judges were asked to sit on the chairs at the front of the room. Very quickly, four people volunteered to be judges, and more reluctantly, 3 people volunteered to be performers. Just as we were about to attempt to cajole a fourth person into being a performer, Gary entered the room and was volunteered as the fourth performer. Without even a chance to settle into the class, Gary was asked to clap.

Performer One (Gary)

As the first performer, Gary was given no directions when asked to clap. Gary sat in a chair and clapped. Once the clapping was complete, Gary was asked to leave the room. The judges assessed Gary's clapping by assigning it a score, from 1 to 4, with 4 representing great clapping and the 1 representing less-than-adequate clapping. The judges were not allowed to consult with one another and were instructed to write their scores on a piece of paper. Gary was then invited back into the room. When he entered the room, we were all solemn. He returned to his chair, receiving no feedback about his clapping. At the same time, the audience did not know the marks that the individual judges had awarded to Gary's clapping.

Performer Two (Sandra)

Sandra was also asked to clap and given no directions. Sandra added some rhythm to her clapping and was then asked to leave the room. In Sandra's absence, the judges were asked to consult with one another to establish their assessment of Sandra's clapping. Sandra reentered the room and was told her scores as well as judges' rationale for the scores.

Performer Three (Malgorzata)

This time, before asking the performer to clap, the judges were asked to discuss and to reach some agreement about the scoring criteria. The audience was then invited to comment and provide feedback to the judges. The judges established the context in which Malgorzata was to clap: a baseball game where her favourite team was winning. Malgorzata was then asked to clap. Malgorzata added a bounce and movement to her clapping. Following the clapping, Malgorzata was permitted to remain in the room while the judges consulted about their scores. The judges then shared their scores and accompanying rationale with Malgorzata.

Performer Four (Jon)

Jon was not only allowed to hear the judges' criteria, he was also permitted to identify any that he would prefer not to be applied to his clapping, to add new criteria of his own, and to define the context in which his clapping would occur—Jon chose a baseball game. Jon's hand was bound because "his dog had bitten him that morning as he left for the clapping exam." He asked for a deferred exam. When that was refused, Jon asked that the judge's take this into account in assessing his clapping. In particular, because of his injury, he would not be able to apply pressure to the affected hand, so the noise level of his clapping should not be used as a measure of enthusiasm. Jon was then asked to clap. Jon clapped the arm of his injured hand with the healthy hand, jumping and moving about energetically, shouting loudly and lifting both arms high in the air. Once the performance was finished, Jon, like Malgorzata, stayed in the room to listen to the discussion of his performance but this time Jon was allowed to participate in the discussion of how he met the criteria.

Performer's Reactions

Performers were asked to describe how they felt about the activity. Lengthy discussions ensued on various themes emerging from the experience.

Gary

> When I walked into the room and you fingered me, I said, "Oh good, another opportunity to embarrass myself." I didn't mind the task, I had confidence that I could clap. I felt good about my clapping, I knew how to do it. But then afterwards when I came back in and with the silence that followed, I started to feel progressively worse, I felt marginalized. And if you want to think about it, because nobody comments on negative thoughts, I was experiencing clapping anxiety. I will never clap again, and I will avoid courses that involve clapping. I'm not getting into any occupation with clapping either.
>
> Judge 1: While you were clapping, were you challenged?
>
> I wasn't before this morning, but I'm on the road now.
>
> Judge 2: Do you think that your impression will change once the judges indicate your score?
>
> No, I feel nothing but dread for what you guys are going to say. Like I saw all those 4's and I saw the enthusiasm [of the other clappers]. I don't have confidence that you are going to do the same thing for me because I don't think I was nearly as enthusiastic. I don't think I smiled when I clapped. I was sitting when I clapped. I feel quite inferior to all of the others.
>
> Woman: What was on your mind before you clapped? Did you think of any event?
>
> Not particularly, it was a generic clap. I'm starting to think, you know, it could have been quite inappropriate the way I clapped, after I saw all these other people do it in a more intelligent way. I was blind-sided. It was like being hit by a car. You know, go from zero to highway speed in zero seconds. I didn't like it at all.
>
> Man: Kind of like a surprise quiz?

Yeah. Like I came in thinking this is going to be a good day. And in 5 seconds, the anti-Midas touch, turning gold into dung.

Man: How could we have changed that?

Through teaching. Another venue. Different classmates, because you saw it all hang out. You all just sat there saying I'm glad it's him and not me.

Man: I don't think it was personal. It wasn't personal.

Yeah. But I mean the non-personal and the lack of community. I didn't, I really didn't feel in a very light place.

Sandra

Well I think I'm in a different position. 'Cause I'm one who volunteered to be a performer. To me it's all a game anyway. So that was my answer. The thing that most disturbed me was the two judges on this end; who did they think they were? They made the assumption that I knew I was going to be asked to clap. And it hadn't even crossed my mind actually. I had no idea what I was going to be asked to do. Just because Gary was asked to clap, I didn't know that I was going to be asked to clap. I really didn't think I was. So I was just willing to go and didn't care what happened and I didn't really—well I did care. I was curious, not so much cared. I was curious to see what the judges would say. They gave me a low mark. I was actually sort of hoping one would give a 1 and one would give a 4, just because I'd be really curious to know why.

Woman: Did you wonder why you were out in the hall so much longer than Gary?

I just figured something was going on. I was looking at the artwork out there. I was actually wishing there was more stuff up on the walls to look at. There was something else, I remember walking up and down a couple of times thinking—like we've all done after a test—why didn't I think of that, why didn't I do that? I did that for a minute and then I just looked at the artwork.

Man: I would just be curious to know what you thought of Gary's comments.

I can sympathize with Gary's feelings, but at the same time I think I would rather have been Gary than the judges. Because they really had a tough task. I think the judges have been put more on the spot in many ways than maybe I was. And I do remember thinking, in context of the fact that I volunteered to do this and I hate games, I mean I personally had the confidence to do that. I had the confidence to say what's the game plan, I'll play the game, I'll do it and not. And the confidence to feel that regardless of the outcome. And get on with my day, and my week, and play the next game, and make sure that I know what I'm doing. I'm quite aware that would be not what most students are like because of that confidence to handle whatever comes my way and do the best I can and not worry about what anyone else thinks.

Woman: I was just struck by what you said about the judges being more on the spot than the performers.

And I remember thinking before we got going, that maybe this isn't a fun performance, maybe we are going to have to do some mathematics and we are going to be like a panel, and you have to do it fast. And I remember thinking, eegads, now we are going to get like the spelling bee of mathematics. Well I don't care if it takes me 2 seconds or 10 seconds and the teacher is going to make me do it in 1 second. Well I did have that anxiety when I thought it was going to be math.

Malgorzata

Like Gary, I was taught to be a good performer. So, I saw Gary and I was a bit surprised by the task and I sensed that Gary was too. I wasn't sure that Gary would be judged on just the task. Because sometimes for me his clapping was rather more of a job than a real performance. I thought that if you think seriously that you are being judged you will probably be stronger. I didn't think that Sandra would be asked to do the same thing. So then when I heard the judges saying that she should, I felt if was unfair because I didn't expect that she would have to do this. So, why did the judges assume she would expect it? I thought it was a bit unjust and I deduced that the judges were not objective and they had criteria which were not quite what they said. Then there was all this discussion about criteria which actually made it worse because I felt Sandra did an excellent job and I couldn't understand why she didn't get a 4. So I knew that I had to do much more than an excellent job to get more than Sandra did. So the question was then to concentrate more on understanding the criteria and the judges' interpretation of the criteria, than on the task itself. So I will tell you that on the one hand setting the criteria may make it easier, but only if you can test it. How do I know what I can do with respect to the criteria if I can't test it? On the other hand it might be difficult because then you constantly try to fulfill the criteria rather than to do the best job you can.

Judge 2: You wanted to do better than Sandra?

Oh, of course. I didn't doubt the judges, but I couldn't quite tell if the movement would actually help. I knew that when I just clap because I wanted to clap, I don't think about rhythm. So I think knowing the criteria made the job more difficult because then you look at all those additional things like the volume and the number of claps, rather than just the task itself. So I would say that our criteria sometimes actually creates more challenges for students.

Florence: Tell us a bit more about the context, because at that point there were a lot of comments about context.

Well, I truly don't enjoy team sports. Twice I went with my husband and my son to a baseball game, but it left me cold, I couldn't relate to it. I'm totally opposed to team sports. On the other hand, the judges gave the score, but I was not sure they had enough experience of my context to be able relate to my context. One of the things I feel most strongly about is being given criteria without being able to practice to see whether you can fulfill the criteria. Because the context itself was a criteria. The context helps because then I can concentrate on how well I'm doing in this context. However having the criteria on top of the context really forced me to think of all the additional things rather than the task itself. And I knew that I had to put myself in the context. Because I knew that I had to be aware of what goes on in that experience.

Jim: Just going back to the beginning of what you said and thinking of it in terms of a classroom situation, do you think then that what you were doing was what the judges wanted rather than learning how to clap? So the important thing was the test not the learning of anything.

Yes that's right.

Jim: Which is a danger then. So, of course, we are back to if it's not on a test, I'm not going to do it. And if it is on the test, how is it going to be tested, and that's what I'm going to do. So the criteria tells you what the test is going to be. And that's all you are going to learn so that takes us full circle back to a bad learning situation.

Yes. Being given the criteria just before doing the task does not allow you to learn because then you just concentrate on fulfilling the criteria. If I had the time to practice it would be a learning experience because then I would learn something about what I'm doing and could reflect on what I could do to improve.

Jon

I really had the advantage of being the last person to go. Seeing everybody else. I thought I was getting a good deal. But when I thought up this disability I thought of doing it. So I felt very confident, but I felt disappointed that the judges didn't believe my excuses. But I guess most of what my feelings were about is not related to me performing or to being a student. They were related to me in my real life as a teacher. And what I am doing to my students has been illustrated here today. And you asked a question earlier about whether we've ever been put in this position as students. What about when we put our students in this position? I feel like I should go back to school and apologize to all my students.

Judge 4: Can I just explain quickly? It just flashed into my mind, and I'm almost certain, that the reason I felt that I couldn't believe you totally was because just recently my professional judgment has been questioned. I had to make an instant decision in similar circumstances. And there I am, and I made a judgement and yes, I believed what the student told me. Now, I'm questioning my professional judgment and in a literal sense, my professional life is on the line. So when I saw you faking it, I couldn't believe you.

DISCUSSION

Interweaved throughout the discussion of each performer's reactions to the clapping activity were a variety of observations, many touching on issues raised in the first day's brainstorming. Following is a sampling of the main areas.

How safe is the environment for students when they are being assessed?

Man: I'd be curious to know how Gary felt when he said he felt really bad coming in and then seeing all the others clap. If he had been the only person, and had been taught to clap, and he clapped and went outside and the judges give him a mark, come back in, and then go on to a different activity – would that have been a less negative situation?

Gary: No. It's like a stick prodding the jellyfish. And the stick really was the teacher and I was the jellyfish. Never in my life do I clap because some adult says okay we are going to judge your clap. I felt like a specimen. I didn't feel like a person that was being respected.

What are some of the constraints a teacher faces in assessment?

Don: The question of performance under a certain set of constraints. What I noticed when this activity was going on, and it was probably largely facetious, but I wondered whether there was an undercurrent of reality: the marks gotta be in tomorrow. Is it permissible; an incomplete, deferred or a medical deferred? I don't know how it works in the school system? But you know I was a little struck that nobody voiced it anyway.

What does it mean to be open with the criteria prior to assessment?

Judge 3: After the discussion I can now see that it was really unfair to evaluate Gary without even knowing the criteria.

Judge 2: I had given a 3, because I had a sense that Gary was struggling and that he knew the task and could do the task. Everything he was asked to do, he had done. But maybe there was more that could have been done, but I had to define what more must be.

Judge 1: I agree with that too. What I had characterized at the time is not much enthusiasm.

Gary: See I didn't even think enthusiasm in a clap. I didn't associate it with enthusiasm, with clapping. I heard clap so I clapped.

Judge 4: My gut reason in doing that was, I'm glad it wasn't me. I wouldn't want to walk in the room and suddenly be told to do something. And I think that in that context , and given the information that Gary was given at that time, I thought he did a damn good job. And it was 9 claps, I remember that.

What assumptions do we make about what our students' know?

Judge 2: You criticized the judges for ...?

Sandra: Making assumptions about me.

Judge 2: But truly though, once you were told in class, you had the experience of Gary that you could immediately turn to and you would reflect momentarily on his performance and think okay what does that mean for me. Could I say that?

Sandra: I don't think I had really thought of Gary. No. Because I didn't know, it was only an instant. I'm asked to clap so I hadn't thought of it. It didn't cross my mind that I'd be asked to clap.

Woman: But that was the only thing that really disturbed you?

Sandra: It was the judges making assumptions about what I knew. That I knew, that I had a preconceived idea, that I was prepared for what I was to do, and I wasn't. I was only prepared in that the whole thing to me was a game anyway.

As teachers, or judges, have we clarified the criteria?

Judge 1: We didn't know the criteria. We hadn't talked about it before. And even after Gary had done his and Sandra was next, I'm not sure she would have known because I wasn't sure whether we were looking at a pattern. And we had seen Gary do it. In a sense his pattern was just the same movement over and over again. Whether we were then looking for patterns? And you would come in and do something, and what you were trying to do—one, one two, one two three, one two three four—and that we were trying to determine the kinds of patterns you might clap? And so after our discussion we weren't even really any closer to knowing whether that was part of what our goal was.

Judge 2: If we look at the range of scores though from the first one with Gary from 2 to 3 to 3.75. After that we got much closer until by the time we were at the last couple we were almost at the same number. We were starting to get very close and I'm sure with a couple more experiences and a little discussion, even though we hadn't defined for you what we meant by our criteria, we were getting closer together on what we thought.

Do we know what a good performance looks like?

Gary: There is one thing that we haven't talked about, that I noticed as a person involved in mathematics education and as a student who did some clapping. It was the way the judges reported to the performers and this was when they were giving 3's, 3.5's, 3.75's. There were a whole lot of scores between 3 and 5. And I thought, hey that's pretty good. And yet, half of the time, half of the statements shared were of a negative nature. To me, there was a disproportionate number of negative rather than positive comments. Which then as a student, made me think, well obviously I was worse than those people and if those are the kind of negative comments they got, I'm really going to be hosed when they pull them out and start saying things to me. But I'm more interested in the assumptions that were made on what levels 3 and 4 were supposed to be. Because I don't think, when we started shoveling in criteria into those levels, we ever really explicitly discussed what the generic meanings were for those levels, and what kinds of descriptors would be used and what student performance would look like at each level.

Judge 1: It seems to me that we're talking about the fact that the comments were negative. That's one way of looking at grading, particularly when the criteria aren't terribly explicit; you have some sort of ideal performance in mind that's really where you would like to see people. And that everybody can do, and in a sense you see that as what each person should be striving for. But if they don't quite make it then you end up with negative comments. But you still have to see that as where they are aiming, in other words it's a subtractive principle. The other way is to start at the bottom and say everybody starts with a zero, or whatever your bottom score is and you add up the pieces as they go. It's easier to say positive things, but you may come out with a lower grade than if you think of a person as at the top first and maybe taking a little off where you say negative things.

Judge 2: I had initially written down a 4 for Sandra, because it was different in the sense that she was doing different things than what Gary had done. Then we went to our discussion and it was at that point when we started to say that we were using 3's and 4's in a sense that it was a 1 through 4 range. And so then I got a sense that if I was to say a 3 or a 4, a 4 might be the top mark you could get, that would be the best performance we would ever see today, rather than at the highest level. So that's the differences in my thinking once we started talking about the other levels, that the 4 was the maximum you could get. And that's why I went for a 3.29, because I knew there was still 2 more people and I had to have some room for the other performers.

Judge 3: So you could get a 4, but it didn't have to be perfect. There was a range. But then some of my fellow judges went to 3.5's and I tried again to make some sense out of how can we have a 3.5 and a level 1, 2, 3, and 4. So for me, we switched back and forth between the levels. Those who are saying 3.5 may have been thinking 4 was the maximum. To me a 4 is a level that could be achieved. And so I'm not sure we ever unraveled that until the very last one and we still had some 3.5's.

Is the level in the task or in the performance?

Judge 3: Well now that I have seen the 4 models, I'd like to go back to the original task just to clap. You did not describe what you wanted to see, you just said clap. And based on that, actually I would change my mark for Gary to a 4 because he did the task he was asked. We didn't have any criteria at all, we just wanted him to clap. It's like asking him what is 2 + 2 and he gives 4. Then when we get to the last performer, Jon, he knows what we are expecting, he is able to communicate with us before he performs. Do the questions change as we see students work? Now it's not just 2 + 2 = 4, we are asking them to show us a full application of the principle 2 + 2 = 4. And so therefore the criteria for judging Jon are more global. Whereas his is just clapping. So I would change Gary's mark to 4 because he clapped, he did what he was asked to do. So there shouldn't be any emotions here, it's just a task. We didn't require enthusiasm, we didn't tell him that he should show appreciation, it was just a clap.

George: I just have a question about the levels. When we first got started yesterday I asked a question about a student of mine who had 95% on a test. So if Gary claps and that was what was asked, is that a level 4? Or does it depend on the question that is asked? So the question is, is the task of clapping a level 4 task? It may be interesting for us to think about what it looks like to be performing at level 3, 4 work in mathematical thinking so we know we are assessing that.

Judge 1: We developed levels for people as we went along. Those levels indicate certain kinds of expectations which ultimately are not independent of the student's developmental level. In other words, where is the student in this, even in learning to clap? We may want to be subjective in our assessment of a particular individual's performance.

Is there a difference between skills and levels in mathematical thinking?

Judge 2: Some tasks have a range like achieved, not achieved. Others have a range such as 1 to 4. But if I ask someone to clap and the task is only to check and see if they have the skill of clapping then we can't start thinking well, that's a 4, that's not something that should be thought of in a level. It's a yes you can do it—check, no we can't do it—check. And that's achieved or not achieved. And I think that's why the framework was important by the time we got to our third or fourth performer and we could give the framework and what we were looking for. Now there was a sense that there was going to be a range because we were looking for a range of answers. And I think that's the same as many of the mathematics questions we ask. Some of them have an answer and that's it and others allow our students to go off in different directions to explain different things. And we have to distinguish between them.

George: When I think of this task that we had to do, even when we get more complicated at the end, I still don't think it's a task that is worth having 4 levels for. In other words, when I think of mathematical thinking I think, what's a level 4 student? And when I think of that person, I don't think that this is a kind of task that I would do to show me if they are a 4 level or not. For example let's say we are doing factoring techniques, and my test is on factoring techniques, and this student has mastered that. Now the student that gets 100% on that, to me, is not level 4 because I need to know more about how they think about factoring and how they can apply it, how they can see in terms of whether to use it graphically or what it means in terms of equations and so on. I need to assess that as well, to see if they are level 4. So I know we use levels 1, 2, 3, 4, for a variety of purposes, but when I think of mathematical thinking, I think this is not a level 4 task.

Pat: I think that I would agree with that. When we moved into the discussion of Jon's clapping and his redefinition of clapping, we really did move things to a whole different level. I could see thinking of clapping as having multiple levels where level 1 is managing to get your hands together but you don't make a noise. Gary could do that. But he made a noise, so he was level 2 in a sense because he could do it well. But there we're just thinking at the skills level. But then you get to Jon's definition of clapping and you can see a variation in how you might assess a clap that is intended to make the performer feel appreciated.

Is the task appropriate for learning and for demonstrating learning?

Gary: I think that there is something really important and sinister about a lot of what we have done here. And we keep coming back to it. It's this whole business of, what is it that we are trying to get the learner to learn here? And to demonstrate. Really this whole business of being able to show appreciation in an appropriate fashion, is a lot more complex than that simple pile of criteria that we have thrown up on the board for this particular task. If in a real situation someone is giving a talk or singing and I'm talking to somebody else paying no attention to them – well that says something about the appreciation I'm about to show. And then, the song comes to an end and I don't even show any appreciation and the only way in which I'll show appreciation is in a constraint situation in the centre of a room when the judges say clap. I don't even want that as an assessment task. I want a different assessment task. It's related to this whole business of whether it give students the opportunity to demonstrate performance at level 4. I need an assessment task where one of the things I'm going to look at is the sincerity of the appreciation that is shown? Is there an appropriateness in the way they show their appreciation? Did they have to be assisted to clap? Or did they do it under their own volition? Phrases like self-motivated, feeling empty, and things like that are what I'm looking for. I would almost have to embed that clapping task in something much bigger. It would probably be some kind of social interaction, so that the clapping would become only one of a number of things that I would be assessing. It's only when it's embedded with a number of other things to assess, that it starts to take on the significance that it should. And this is what we have been raving against, writing a curriculum that is too damn narrow, excuse the language, and then taking good curriculum and interpreting it too damn narrowly. And then having corresponding learning activities that are too narrow and assessment activities that are too narrow or not appropriately authentic enough. And I think that this task is just a wonderful vehicle for getting at those issues.

CONCLUSION

In the final session, group members generated and prioritized a list of issues for further exploration. The topics of greatest interest were elaborated in small group discussion and are summarised briefly below.

The Role of Assessment in Enabling Students to Demonstrate Their Mathematical Thinking

The following list, though not exhaustive, describes skills and attributes of students who are successful in mathematical thinking:

- creative;
- able to analyze problems;
- able to discover things for themselves (constructivism);
- see relationships;
- see the "whole" picture;

- know how to build the picture;
- have good number sense.

A key task is to design and implement an assessment process by which students are able to demonstrate that they possess these skills and attributes and through which teachers can recognise that their students possess them. Essential is an environment which nurtures and supports mathematical thinking by:

- providing freedom, opportunity and time;
- meaningful, rich, open questions;
- building students' confidence;
- providing challenge; and
- focusing on understanding.

The challenges teachers confront in providing this environment are many and range from the political to the personal. Pressure from parents who understand success as high grades on routine tests is one thing. More challenging is justifying what you are doing to parents who have never engaged in mathematical thinking themselves. Worse still, is recognising mathematical thinking in your students in this circumstance.

The Role of Context in Assessment

A variety of issues and questions were explored by this group:

- there is a place for evaluating mathematical skills in a non-contextual manner;
- there needs to be a balance between abstract and contextual problems;
- skills should be assessed in context only if they are taught in context;
- is it more important to assess higher order thinking skills in context than to assess low level knowledge?
- assessment should be fluid and permit students to explore mathematics;
- is there a difference between the use of context as an instructional strategy and as an assessment strategy?
- assessing exploratory skills within contextual situations may limit the exploration;
- some contextual situations are contrived to the detriment of the study of mathematics;
- without a good understanding of the problem, some contexts may mislead the class;
- some contexts are unsuitable, for example, because they do not interest students or they are socially unacceptable.

The Purpose of Assessment

What is the purpose of assessment? What are/should we assess? and why? In exploring these questions, this group focused on systemic issues and ways of assisting students in determining what to do next when exploring a challenging problem.

Curriculum as Defined by the Assessment Tasks

This group was seized with the questions:

- What does the assessment task give the learner the opportunity to demonstrate?
- Is what the assessment activity gives the learner the opportunity to demonstrate what the curriculum indicates a successful learner should be able to demonstrate?
- Is "what the curriculum indicates" a successful learner should be able to demonstrate worthy of pursuit?

Several conclusions were reached, chiefly that assessment tasks should:

- focus on big ideas as well as small;
- allow learners to demonstrate what they have learned and can do; and
- indicate to learners, teachers and others, what is important and valued.

The Role of the Student in Assessment

Where is the student in assessment? Three roles were considered, namely, self-assessment, peer-assessment and assessment by and of groups. This raised several related questions:

- How ready are different groups to assess each other?
- In what circumstances do we accept students' own self-assessment? For example, assess the whole course mark or only part of it.
- Can students' own self assessment be assessed?
- Who generates the criteria for self- and peer-assessment, and how? Before or after seeing the work?
- At what ages/levels can self-assessment be engaged in?

Other Questions

Of course, we concluded with more questions than we had when we began. Several that we did not begin to discuss will provide fuel for future working group discussions:

- What are assessment strategies?

- How can teachers develop special assessment tasks and criteria that do not 'blind' us to authentic mathematical thinking and hence opportunities for assessment?
- How can we use assessment to motivate students?
- In what ways can assessment results be reported?
- Does specifying the assessment criteria always help students to (a) perform better on the test; (b) learn and understand the concepts?

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Working Group B

FROM THEORY TO OBSERVATIONAL DATA (AND BACK AGAIN)

Carolyn Kieran, Université du Québec à Montréal
Jo Towers, University of Calgary

PARTICIPANTS:

Olive Chapman, University of Calgary
Rina Cohen, OISE/University of Toronto
José Guzmán-Hernández, Université du Québec à Montréal (and CINVESTAV, Mexico)
Dave Hewitt, University of Birmingham, UK
Klaus Hoechsmann, University of British Columbia
Carolyn Kieran, Université du Québec à Montréal
Lloyd Lawrence, University of Manitoba
John Mason, Open University, UK
Joyce Mgombelo, University of Alberta
Cynthia Nicol, University of British Columbia
Geoff Roulet, Queen's University
Elaine Simmt, University of Alberta
Jo Towers, University of Calgary
Vicki Zack, St. George's School, Montréal

INTRODUCTION

"The phenomenon is the children working"
"The phenomenon is the videotape of the children working"
"The phenomenon is the story being told about the children working"
"Videotape is no good unless you were actually there"
"The videotape is the data"
"Data does not exist independently of the observer"
"Videotape is just videotape until we offer it to a community who decide to make it data"
"I have a relationship with the data"
"I am the data"
"I can see more clearly now how blurred the notion of data is"
"The best we can do is tell stories"
"What I learn from a researcher's analysis is what they are sensitive to"
"We can't see anything unless we have a structure to see it"
"I can only begin to work out my structure when I begin seeing"

How might statements such as these have emerged in a three-day conversation amongst peers? What might have led up to their expression? Those present at the reporting session on the final day of the 1998 Conference might recognize these statements as those read out by all the participants in this Working Group. Placed as they are here, with no preamble and removed from their context, the statements might suggest that our Working Group was a place of conflict, disagreement, and confrontation. As participants as well as leaders of the Group, though, we experienced a far more productive environment. We offer these statements as examples of the breadth of our conversations and hope to reveal in this report how the participants of the Group created an expansive and thought-provoking discussion which embraced these diverse points of view.

A STORY ABOUT VIDEOTAPE

Let us step back, for a moment, and consider how this all began. Our story begins early in 1998 as we began to explore ideas for the Working Group we had been invited to lead. At that time the Working Group had only a theme (suggested by the Executive) for structure— "From Observational Data to Theory." In one of our first e-mail exchanges, we decided that the suggested title didn't capture adequately the relationship between theorizing and data collection and analysis, and so began our conversations about the nature of theorizing and its relationship with data. Soon, the new title for the Working Group was born— "From Theory to Observational Data (And Back Again)." So far so good. However, the problem immediately became "So where do we begin in our Working Group—with theory or with data?" We began to explore possibilities for the structure of the three-day experience and it soon became clear that we needed to find just the right piece of videotape to share with the Group before we could plan how the three sessions might evolve. This became our priority, and we decided to try to obtain copies of the TIMSS (Third International Mathematics and Science Study) videotapes. When we finally tracked down a copy of the CD-ROM version, we were disappointed to discover that they contained only short excerpts of a number of lessons, and little context was provided to situate the clips. We believed that we needed to find a rich data source to support the exploration of a range of issues and to engage the members of the Working Group in nine hours of conversation and investigation. Our search began again. We finally settled on an excerpt featuring two Grade 7 students engaged in problem solving. We hoped that the excerpt, which was about thirty minutes in length and for which we also had copies of the students' written work and full transcripts of their conversation, would support investigation in a number of areas, such as the nature of the students' interaction patterns or the growth of their mathematical understanding. As we will recount later, though, all our agonizing over video excerpts was to prove unnecessary.

A STORY ABOUT THEORIZING

Now that we had just the right piece of videotape, we turned to the task of planning the flow of activities for the Group. After lengthy e-mail exchanges we decided to begin the first day by having the participants introduce themselves and talk a little about the theoretical frames they use when analyzing their own data or conducting their own research, and then to broaden the discussion to consider the question "What do we all mean by 'theorizing'?" before introducing the chosen video excerpt sometime towards the end of the first session. Day One began just as we'd hoped, with an enthusiastic and diverse group sharing stories of their research interests. Many of the stories included passing reference to recognized philosophical frameworks such as social constructivism, enactivism, or phenomenology. As the conversation broadened to explore the issue of what we all meant by 'theorizing,' several participants explained their reluctance to affiliate themselves with a particular strand of thought such as, for instance, constructivism. One member articulated this in the following way:

> I don't want to say that I belong to [that] club. I don't want to define myself in that way. And so, social constructivism? Yeah. OK. Fine. Radical constructivism? Fine. Behaviorism? Yeah, some of that as well. ... And actually there's quite a lot around that's quite helpful depending upon what it is that I'm looking at. ... But what I feel like saying, what I refuse to do is say 'I'm this'.

This sentiment was echoed by another participant a few moments later:

> The problem I have is not with identifying myself with some perspective of thinking, but it's the community wanting to keep me there. I would like to be a radical constructivist today, an enactivist tomorrow, a social constructivist the next day, because ... what it is that I'm interested in at that particular moment would dictate where I would like to be.

As the discussion about what is meant by 'theorizing' broadened, one participant remarked on the need to take the discussion a step backwards:

> I think we need to talk about what we think theory is before I can talk about how I use it.

Another participant responded by indicating that:

> Some of us might have to [talk about] both together. In order to talk about theory [we need to] give it some context [by] describing how it's used.

The discussion continued as Group members attempted to articulate their notions of theory and what it means to theorize. Comments included:

> Maybe theory is just something that arrives in what you do, that it isn't 'out there', a framework.

And:

> I guess my theoretical framework is there for me to ignore it.

Some participants described aspects of their own research work to explicate how they used theoretical frameworks in structuring their practice. We then offered a paraphrased excerpt from Cobb and Whitenack's (1996) paper on data analysis using videotape. The extract briefly outlined Glaser and Strauss's grounded theory method and emphasized Glaser and Strauss's view of the inextricability of the development of theoretical constructs and the process of data collection and analysis. One participant reflected on the extract by drawing a distinction between the 'grounded theory' discussed in the paragraph and a 'theory' like constructivism. He pointed out that the Group had been using the word 'theory' on (at least) two levels. Other participants responded by trying to differentiate between theories that structure the questions they pose, and theories that structure the ways in which they seek answers to those questions.

It was increasingly becoming clear to the Group that trying to talk about theory in the absence of data was problematic. One participant articulated this sentiment in the following way:

> I think it's interesting how hard it is to talk about something like a theoretical framework or theory before having ... shared experiences that help foster the conversation. I'm waiting for the shared experience so that I can have some examples to use. I'm stuck without examples.

This seemed to us to be the perfect moment to introduce the video extract we had chosen. As preparation, we first explained the context of the research, and handed out copies of the problems posed to the two Grade 7 students featured on the tape. Approximately fifteen minutes of the videotape was played to the Group, after which we asked the Group to share with us the aspects of the videotape to which their attention had been drawn. There was a variety of first impressions and features that had captured participants' interest, including aspects of school culture that one participant suggested were embedded in the students' problem-solving techniques, the influence that a re-focusing of the camera lens during the episode had had on another Working Group participant's focus of attention, issues surrounding the interaction between the two students, particularly in terms of their collaboration processes, and the mathematical language used by the two students.

After initial discussion of these and other issues, the Group was split into two sub-groups, and one group moved to a separate room with a second copy of the videotape, to facilitate a more thorough investigation of a number of these issues.

A STORY ABOUT THE MEANING OF DATA

When the Group met on the second day to continue discussion about what they had been seeing in the videotape, it did not take long before a rather substantial issue arose, that is, what do we mean by the word 'data'? Is the videotape data? If not, what is the data? (In this report, the word 'data' is used in its singular form—the form most often used by Working Group participants.)

A provocative entry into this conversation occurred with one participant's statement:

> I have a theoretical position, which is that there is no event. The event consists of, for me, a collection of stories that people tell, which accumulate and accrete around this particular bit of videotape.

If, as this participant argued, the event is the collection of stories, then how does such a position accommodate the notion of data and its existence? From that moment on, discussion moved in and out of what we thought we meant by 'data'. The perspective offered above was contrasted with others that included, for example, "seeing things in the videotape" or "referring to the data out there." Someone suggested that the polarity of the various positions could be crystallized with a question that is often asked by constructivists or by radical constructivists: "Where does reality reside?".

As the discussion continued, a participant stated that, in her own research, she found herself "returning to the videotapes to generate more data" and that, in so doing, former interpretations subsequently became data. She further suggested that, for her, it could be problematic to try to tell a story about another's videotapes because "you weren't there." The videotape and transcript that another provides are like notes of some event that occurred when you were not present. "It seems that you have to be there," she offered.

Another participant then put forward the idea that, whatever the phenomenon of interest, she could collect data on it by a variety of means. Thus, it was claimed that we could look at the same data obtained through different data sources. In other words, there was an object, called data, and rather than referring to the videotape as data, we might simply call it a data source. This led another participant to move in a somewhat different direction and voice how she felt that the relationship between a researcher and her data is a constantly changing one. Thus, the issue of whether the data remains unchanged—an implicit assumption of some of the earlier remarks—is a moot point, for it is never possible to return to it and to see it as it was previously. Our relationship to it has changed, as a result of interpretations we have brought to it. This opinion was echoed by some, but not by others who maintained that it was perhaps simply our sensitivity to the object called data that was changing rather than the data itself.

Considerable zigzagging continued to occur between the idea of researcher and data as one, and the separation of the two. For example, one participant advanced the question of how, if a researcher and her data are one, the various members of a given research team who look at the same piece of videotape can ever hope to negotiate commonality with respect to what is going on; in other words, how does the oneness become shared. [The same dilemma is one to which constructivism has been at odds to respond, that is, how do we come to have shared meanings if all sense-making is individual.] The question was,

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in fact, finessed by one participant's response that it is the collective resonance by a group of researchers with respect to what they are analyzing that constitutes the data (or that is equivalent to the data).

One participant returned again to the related claim that the data is one step removed from the videotape, which provoked once more the question of what she meant by the data. At that point, someone suggested that we might save ourselves some grief by eliminating the word data from our discussions and another followed with the reminder that, in the hermeneutic tradition, 'text' is used rather than 'data'. But the word data would not go away.

One participant then shared why she had problems defining the word data:

> I'm not even sure that I want to specify what I think is data in my research. I'm not sure that it's helpful for me to say clearly whether the videotape is data or whether I'm the data or whether my relationship with the tape is the data. ... I'm studying my own practice, my own classroom. ... And then when I'm looking at that videotape later, I find it really hard to decide what it is that is the data. Is it the videotape that I'm seeing of this pair of students? But I know when I'm watching it I experience all kinds of things other than that tape, because I was in the room at the time. ... So there's all kinds of other influences on what becomes the data. I guess I'm asking how helpful it is that we actually do decide to call one thing the data.

Another participant spoke of the entries that she makes in her journals when she does her research observations and pointed out that she uses these journal entries as her data. This statement led to an attempt by another participant to distinguish between what a linguistic analyst might do with those journals (i.e., use the journals themselves as the phenomenon of study—as the data) and what a mathematics education researcher might search for in those same journal entries. He continued with the following:

> What data summons up for me is a world of significance—the world of significance is as a phenomenon which is in itself that which is being worked on. So, if the videotape is THE data, then the phenomenon is the sequence of phosphorescences that take place and we can look at that in all sorts of ways—from film techniques, etc., etc. But somehow we're trying to do more. We're trying to find meaning within the content of it, not as simply a phenomenon in itself. ... At a first level of analysis, we are trying to locate the phenomenon.

Soon afterward, it seemed that a fatal blow to the slowly crumbling edifice of "videotape as data" was struck when one participant suggested that the same videotape and transcript could be given to, say, linguists, sociologists, psychologists, or mathematics education researchers. Each community of practice would focus on different aspects presented in the tape and transcript with the aim of arriving at consensual resonance within their own particular sphere of experience. The data for each group would be different. Thus, this example served to clarify the notion that the videotape is just a videotape until a community of practice looks at it and says that some particular thing is going to be data. It (whatever the it is) becomes data when we bring something to it; it becomes data at the moment when we make it data.

The story that has just been told did not proceed in a purely linear fashion. As one idea moved to the foreground, another moved into the shadows, only to re-emerge at a later time. However, just as the changed relationship between researcher and data makes it impossible to 'see' the data as it was seen initially—a notion advanced by one participant during our discussions--so it was equally problematic for these story-tellers to try to tell the story with the same voice in which it actually unfolded. Sensitivities that grew out of the three days of going in and out of the meaning of data make it quite unrealistic to attempt to go back and narrate events as they were prior to the dawning of new awarenesses. As well, there is no real ending to this story. Even though certain issues related to the meaning of the word 'data' have moved out of the shade into the light, others remain a blur.

A STORY ABOUT SURPRISES

Though the story we have just related began early on the second day, it continued throughout the remainder of the Group's time together. So intense was the discussion that the Group never returned to the videotape, a fact that surprised us as Group leaders very much. One viewing of fifteen minutes of tape, and a few minutes in which the sub-groups re-viewed parts of that tape, had been sufficient to generate two days of discussion. Though we (the Group leaders) had agonized for many weeks over the precise nature of the tape we should offer to the Group for analysis, our deliberations had been in vain. Or had they? During our preparation of this report, we have had cause to reflect on the value of the tape we used. In discussing the unfolding of the conversations that occurred in those last two days, we now believe that despite our earlier worries almost any tape would have sufficed. What seemed to be critical was that the Working Group had, as one participant put it, a "shared experience." Fifteen minutes of videotape had served not only as an 'example' to stimulate conversation, but also as a gathering place for ideas. Though the Group's contact with the videotape itself was brief, we believe it was necessary. What participants had seen and heard on the videotape, and their recollections and re-tellings of those events (if for a moment we can have permission to call them 'events'), became locations for the exploration of shared meanings. Participants referred frequently, at least initially, to specific aspects of what they had understood to be happening on the tape, and these explanations helped us all to understand the different positions that participants were taking with respect to both the nature of data and that of theory. In this way, strong statements of position, some of which are at the start of this report, had meaning for those of us in the room on those days. A reference to "the videotape" began to mean not only the specific videotape that we had all watched, but also other videotapes which we may have seen in the past, and ones we might see in the future, and by extension, other forms of recording material.

A second aspect of our work with this Group that surprised and excited us was the way in which the conversation, though wide in scope and great in depth, never strayed far from the orienting theme of the Working Group. We had anticipated that some participants might become so caught up in investigating a certain aspect of the happenings captured on the videotape, that that sub-group might wander far from the theme of the Working Group, and be left with little time to think deeply about the issues we have described in this report. As we have already indicated, though, the participants' contact with the videotape was (by mutual agreement) brief, and, in fact, the sub-groups were never re-formed after the first day. Instead, we participated in the conversation as a whole group, feeling no need for sub-division, or further stimulus from the videotape. The theme that we traced, best described by our title "From Theory to Observational Data (And Back Again)," provoked a zigzagging of discussion from theory to data to theory to data. It seemed that each time we returned to one or the other of these two locations of conversation, the nature of the 'object', be it theory or data, had changed. Neither of these two elements of the conversation seemed static, and each was defined and re-defined, shaped and re-shaped, explored, examined, and investigated many times. Each position with respect to the nature of theorizing or the nature of data (and there were many more than we have been able to present here) was critiqued; each argument probed for insights. Participants' positions with respect to what is data, or what might a theory be, changed, sometimes subtly, sometimes more radically. And all this while, the theme of the Working Group was closely being traced.

We hope that the stories we have shared here reveal something of the complexity of the conversations that occurred, and the nature of the issues with which this Group grappled, over the three days. If the stories told here seem to have lost something in the telling for those who were not members of this Group, we nevertheless hope that this report will be revealing in another sense. As one participant suggested, "What I learn from a researcher's analysis is what they are sensitive to." Clearly, what is reported here are the statements that we heard, accentuated by our understandings, augmented by our interpretations, and shaped by our biases. We make no apologies for this fact, but acknowledge that what we have presented here is a story by two people representing a conversation among fourteen. It is, though, a story that we hope might provoke further reflection about data and theorizing, and the relationship between the two. After all, to echo the words of one participant, "The best we can do is tell stories."

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Working Group C

BRINGING ETHNOMATHEMATICS INTO THE CLASSROOM IN A MEANINGFUL WAY

Kgomotso Gertrude Garegae-Garekwe, University of Manitoba, Winnipeg

PARTICIPANTS:

Ann Anderson, University of British Columbia
Patricia Barnson, University of Manitoba
Nadine Bednarz, Université du Québec á Montréal
Mary Crowley, Mount Saint Vincent University
Sandy Dawson, Simon Fraser University
Kgomotso Garegae-Garekwe, University of Manitoba (co-leader)
Linda Gattuso, Université du Québec á Montréal
Susan Gerofsky, Simon Fraser University
William Higginson, Queen's University
Marty Hoffman, Queens College (co-leader)
Ann Kajander, Lakehead University
Tom Kieren, University of Alberta
Eric Muller, Brock University
Eileen Phillips, Vancouver School Board
David Pimm, Michigan State University
David Reid*, Memorial University of Newfoundland
David Wheeler, Vancouver

* David Reid helped with graphics

INTRODUCTION

The following questions were prepared to direct our discussions:

1). What benefits can we get from bringing ethnomathematics and the history of mathematics into the classroom?

2). What aspects of ethnomathematics and the history of mathematics can we bring into the classroom?

3). Which topics from mathematics curriculum can be enhanced by using concepts from ethnomathematics and the history of mathcmatics?

4). Can we use technology to explore ethnomathematics? How?

However, the questions were not necessarily discussed in this sequence. Rather, each participant was given an opportunity to define or describe what ethnomathematics meant to them. It was evident that participants in this group had different notions about the term ethnomathematics. Of course, having diverse understandings about the topic led to numerous questions, some of which we did not attempt to answer. Basically, the discussions were centered on six themes, and these are summarized below.

DISCUSSIONS

This summary was not written in chronological order. The rapporteur tried to weave the discussions together, so that the proceedings make sense to the reader.

What is ethnomathematics?

In discussing the meaning of ethnomathematics, we realized that the term meant different things to different people. To some, ethnomathematics is the mathematics as experienced in everyday situations as opposed to the mathematics taught in schools. To others, it is an approach to teaching and learning mathematics. It seemed to me that each person's notion of the term was determined more or less by their experiences with ethnomathematics, which was further influenced by the level at which the person teaches; viz., Junior High, Senior High, College, or University. Moreover, each person's understanding of the term was influenced by their profession: a mathematician or a mathematics educator.

The idea of ethnomathematics as an approach can be drawn from Ubiratan D'Ambrosio's view that ethnomathematics is a programme. "This is NOT a theory, or a concept, or a method, but rather it IS a programme..." (Hoffman and D'Ambrosio 1991: 34). His definition of ethnomathematics "helps [one to] focus on the activities involved, connecting the development of human cultures..." (Frankenstein's comment in Hoffman and D'Ambrosio 1991: 34). The focus here was on activities: Does it mean then, that the process (doing or being involved in activities) is the ethnomathematics, and the product is mathematics? If so, why then do we talk of ethnomathematics as something new and different from mathematics? Are ethnomathematics and mathematics parallel bodies of knowledge with similar functions?

Again, the report from the small group that was assigned to discuss the definition of ethnomathematics left us with more doubts and questions than we came with. Mathematics has a structure. That is, there are criteria by which one can determine if a piece of work is mathematics. In contrast to this, there are no criteria with which to identify a piece of work constituting ethnomathematics. So, what should we call patterns and designs (generated from multiple reflections) we find in other cultures' activities? (Eglash, 1998). What should we call counting, computations, ratios, fractions, estimation, and mathematical applications we find in other cultures? (Bockarie, 1993). Should we look for another term that would not fail the criterion of structure? That is to say, is the label problematic?

There emerged a concern about the tag 'ethno' prefixed to mathematics. What is the tag 'ethno' signifying? Webster's Third New International Dictionary (1986) gives two definitions for the word ethno: (1) race, people, cultural group; (2) characteristic of or believed by a people, race, or group. Is ethnomathematics then the mathematics that a race or a people believe to be mathematics? Do these "identifiable cultural groups" (D' Ambrosio, 1985, p. 45), the Incas for instance, call the activity of storing information by tying knots (Ascher, 1991), mathematics? Or do the Tchokwe recognize sand drawing as graph theory? Do these groups believe these activities to be mathematics? Or is it the Euro-American researchers who believe these activities to be mathematics: something similar to what is labeled mathematics? In other words, do we mathematize activities in other cultures?

Furthermore, questions were raised as to whether the concept of ethnomathematics differed from that of the anthropology of mathematics, and whether it is another content to be added in the curriculum?

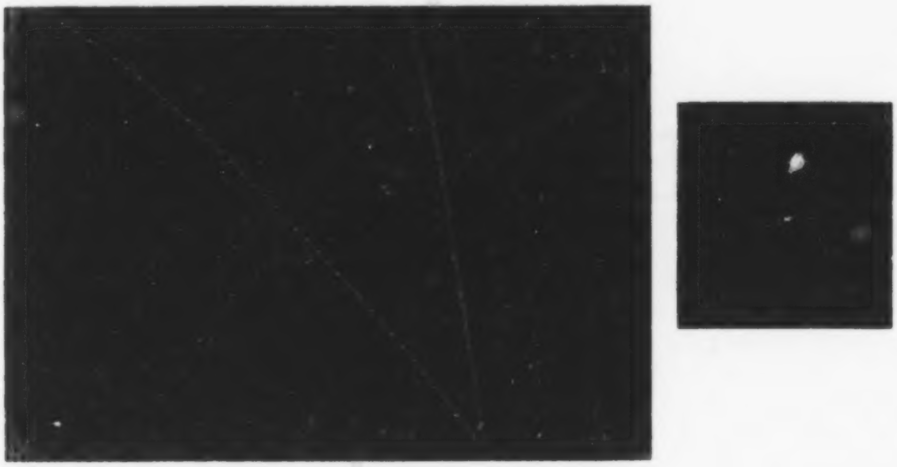
Which is which? Is it bringing ethnomathematics into the classroom or taking school mathematics outside the classroom?

The question of whether ethnomathematics was an approach or another content is pertinent here. Why can't we say we are taking school mathematics from the classroom to other cultures when using their activities to explain a concept to students? Do we want to change the language used (by, e.g., farmers) in trade mathematics? If ethnomathematics is an approach, can we say we are bringing an approach into the classroom? What if we follow D'Ambrosio's view of ethnomathematics as a programme? Can we say we are bringing this programme into the classroom? Doesn't that suggest that ethnomathematics is another content? If indeed, it is another content, do we have a space for it in the curriculum?

Is the label problematic?

Time and again, the group revisited the question of labeling. There was a realization that labeling is problematic: When does a piece of work or an activity become mathematics, and when does it become ethnomathematics? Is it possible that the same piece of work could be labeled differently by different people, say, a mathematician and a farmer? Frieze patterns and quilting designs were brought up as an example of this dilemma. Frieze designs are generated by reflections through 180 degrees over parallel lines. An original figure and its image are reflected repeatedly over subsequent lines. This process produces skirting designs usually found on the cover of books as well as on cultural wall and floor designs. Is the process mathematics, and the product a piece of ethnomathematics, or vice versa? Where do we draw a line?

Below are two pictures showing the same pattern: (a) is a photograph of a tile from the Palace in Sevilla, and (b) was produced by reflections on the sides of a 30-60-90 triangle. Where is mathematics and, where is ethnomathematics?



(a) (b)

Figure 1
A tile pattern from the Palace in Sevilla (a), produced by reflections on the sides of a 30-60-90 triangle (b). Photograph by David Reid, 1996.

It was observed that, if a piece of work does not pass the Western structure, it cannot be labeled mathematics. It is ethnomathematics, meaning the mathematics of the others. The 'others' here, is not exclusively a race group. It could be a profession, such as engineering. But isn't school mathematics someone else's ethnomathematics? For example, the concept of confidence level used in statistics comes from the British cultural practices (Eric Muller's comment). Why then should we use someone's ethnomathematics as a yard stick to determine if a piece of work from other cultures is mathematics or not mathematics?

It may be time to revisit the concept of scientific sieve. We are operating at a point in time where knowledge in the whole world should be explored and used. It should be looked at as equally important, not regarding the other as inferior. During the discussion, a comment was made that some paper folding activities violate mathematical principles. Isn't this one example that should alert mathematicians and mathematics educators to revisit mathematical laws? Is mathematical knowledge infallible? Is mathematics static or dynamic? The discussions on labeling a piece of work as either ethnomathematics or mathematics revealed clearly that we need to re-examine what we mean by mathematics. In this light, we need to examine where and how ethnomathematics fits into the mathematics? In fact, one participant was surprised by the way the structure of mathematics was emphasized, and asked: "Why should we defend mathematics that much?"

What are practical and ethical issues we need to consider when incorporating ethnomathematics into the classroom?

Are we prepared to take on board the cultural aspects of mathematics? Are we willing to venture into the unknown? How is ethnomathematics going to affect our practice? Are we not stealing someone's creativity and using it for our own benefit?

These questions were raised in order to assess the feasibility of incorporating ethnomathematics into the classroom. The questions concern mainly the practical and ethical issues. Practical issues included the standardized curriculum, the assessment, and cultural diversity of class composition. Individual schools standardized mathematics curriculum in order that all students in each grade level receive similar content in a year. In some countries, the syllabus is standardized throughout the country. This is to make sure that students in the country get exposed to the same mathematics. Therefore, incorporating mathematical activities that are local to students may pose a problem: Students in the same country may be exposed to different experiences.

Furthermore, the kind of questions asked in the examinations would influence how teachers, students, and parents could treat ethnomathematics. If examination items do not include materials with ethnomathematics flavor, teachers; students; and parents are likely to ignore, or even reject the use of mathematical ideas from other cultures in schools.

At this juncture, the question of diversity of cultures in the classroom becomes pertinent. Does it matter who is doing mathematics and who is teaching it? Whose ethnomathematics will be used in case of diverse cultured classroom? In case of mixed classes, it is possible that some students may not know the activities (such as games) that are being used in the classroom. As such, the notion of using activities that students are familiar with fails. These questions raised suspicion about the practicability of incorporating ethnomathematics into the classroom.

The last question raised under practical issues concerns the intention and attention—the shift of attention. What are we telling the teacher to do? What is a teacher supposed to be responsible for? Is it to develop local materials for use? Who sets the criteria to develop classroom materials from ethnomathematics? Do teacher schedules have room for more duties?

In addition to practical issues of bringing ethnomathematics into the classrooms, ethical issues were also raised. One of these concerns the validation and reaffirmation of knowledge from other cultures. It was observed that when we bring ethnomathematics into the classroom, it is like saying, the world of mathematicians and mathematics educators recognize cultural practices (other than Euro-American) as valid and useful. Who validates knowledge? How is it validated? Is validation done through the scientific sieve or standard? What about if knowledge from other cultures does not satisfy or pass the scientific sieve? Does it mean it would not be knowledge?

Another ethical concern relates to the use of someone's creativity without acknowledgment. These cultural designs and patterns were made by individuals. Reproducing them for use in the classroom without proper acknowledgment is tantamount to stealing from the designer. But, does this argument hold water? Do we acknowledge those who developed mathematics earlier? What about the formulas we use in the classroom? Do we always mention the name of the person who came out with that formula? Taking the example of common fractions, do teachers acknowledge the Chinese, or the Hindus, or the Arabs? Nevertheless, if the argument is pertinent, the designs could be named after the designer. However, issues raised above warn educators to be more careful and vigilant with regard to what they do with mathematical ideas from other societies.

Why incorporate ethnomathematics into the school curriculum?

If educators want to incorporate ethnomathematics into the classroom, it implies that they have found its usefulness and would like to tap from it. What is lacking in school mathematics that has motivated the mathematics educators to want to incorporate mathematical ideas from other cultures into the classroom? Further, does it mean that teachers were not using children's activities in the classroom before the literature on ethnomathematics? Probably not!

In the course of the discussion, some benefits of using ethnomathematics in the classroom were noted. Ethnomathematics helps teachers to realize that there is more than one way of doing things and, that mathematics has an impact in societies; even if such an impact is different from the teacher's perspective and perception. Moreover, the literature gives several advantages of using mathematical activities from other cultures. Among them is setting the context for teaching and learning (Zaslavasky, 1994; Scott, 1988). Mathematical ideas from other cultures provide a context within which the teaching and learning of mathematics could be enhanced. Mathematics teaching has long been criticized for its abstractness (Shirley, 1995). This kind of teaching makes it difficult for learners, especially at the elementary level, to grasp concepts with ease.

The discussions in this group showed that using activities that students are familiar with make students participate with enthusiasm. One participant reported that her students were active and curious when she incorporated activities that students were familiar with in the classroom[1]. She said: "It was her (the student's) first time to ask why something works". In this participant's lesson, students could relate the concept taught in school to what they already know. They made sense out of the teaching and learning process.

Another participant demonstrated how he uses Egyptian Fractions in his teaching. As the group engaged in the task, there was evidence that using ethnomathematics or the history of mathematics in the classroom widens the horizon of both the teacher and students. They both see that there are different procedures for the same operation, and most importantly that, mathematics is a human creation—it is dynamic, and still developing.

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[1] Presumably this was the case of a class constituted of pupils from the same cultural group.

What further work needs to be done?

Some group members had reservations about using ethnomathematics in the classrooms. They thought that such practice could weaken or dilute the school mathematics. It is possible that such fears are due to unanswered questions concerning the authencity of ethnomathematics, some of which we have raised in this report. An examination of questions such as: "What gap is there between mathematics practice in school and mathematics practice in everyday practice?" may alleviate these fears.

Another concern noted by the group is the fear that researchers (and some of the group members) may have deviated from the pioneer's definition and descriptions of ethnomathematics (and the expectations thereof). An examination of the literature on ethnomathematics, juxtaposed with Ubiratan D'Ambrosio's initial work may detect if we have deviated or not. An interview with him may also be helpful.

Also, there were some reservations as to whether actual academic learning takes place when students are engaged in some of these cultural activities. For example, in the case of playing a rope game: What are they learning when they are jumping over the rope? Are they making some connections with the measurement concept, for instance? Furthermore, do teachers use time wisely during the play time? Isn't there a possibility of spending the whole lesson period on a game without making any mathematical connections? Is it worth it, if such situations are encountered, especially if it occurs more often? Future research, therefore, may examine these practical concerns, and advise curriculum developers and practitioners on how to incorporate ethnomathematics into the school curriculum.

CONCLUSION

Even though the group did not cover all the questions raised by the organizers with regard to the topic, members engaged in germane discussions which provided clues on why implementing an ethnomathematics approach into the classroom seem to be slow. There was a need to understand what we meant by ethnomathematics, as well as the need to be aware of the practical and the ethical issues embedded in the topic. There was, however, a consensus that the teaching of mathematics in the 21st century must differ from what it has been in the past decades.

In summary, the group's discussions provided the groundwork toward formal acceptance or non-acceptance of incorporating ethnomathematics into the curriculum. Most of the issues taken presumptuously, for example, classroom politics (diverse cultures, assessment, the teacher's culture, attitudes of parents, students and teachers), were challenged. Thus, this report is meant to arouse the reader's curiosity about the topic, and become more sensitive than before about using ethnomathematics in the classroom. The report has not covered all the issues that were discussed in depth. I therefore, invite comments and reflections concerning the report and topic in general.

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Working Group D

MATHEMATICAL SOFTWARE FOR THE UNDERGRADUATE CURRICULUM

Frédéric Gourdeau, Université Laval
Michael Monagan, Simon Fraser University
with the assistance of
Joel Hillel, Concordia University

PARTICIPANTS

Sen Campbell, UCI
Alan Cooper, Langara College
Brent Davis, York University
Astrid Defence, Université de Montréal
Luc Demers, Université d'Ottawa
Jean Dionne, Université Laval
Claude Gaulin, Université Laval
Gila Hanna, OISE - University of Toronto
Bernard R. Hodgson, Université Laval
Lesley Lee, Université du Québec à Montréal
Dave Lidstone, Langara College
David McLaughlin, Kwantlen
Ralph Mason, University of Manitoba
Roberta Mura, Université Laval
Medhat Rahim, Lakehead University
Peter Taylor, Queen's University
Jana Trgalova, Concordia University
Rina Zakis, Simon Fraser University

INTRODUCTION

This working group was officially chaired by the two authors of this report, namely Frédéric Gourdeau and Michael Monagan. The third leader, Joel Hillel, graciously agreed to help chair the group on the Sunday (as Michael was unable to attend on that day). His advice and help during our three days of meeting was significant, and we wish to thank him. However, we (Michael and Frédéric) are solely responsible for this report, and any of the good advice given by Joel may well have been lost. So please, do not blame him for any of the outrageous opinions put forth in these pages!

As we were approaching this working group, we were fully aware that the objectives we had laid out were too ambitious. It is probably fair to say that the 18 participants were also fully aware of this fact. However, in making our plan, we had tried to ensure that we would be able to engage in a useful and pertinent experience, and that we would then be able to have a fruitful discussion on some of the impact of software on teaching and the curriculum for undergraduate mathematics.

We feel that this goal was achieved, at least in part, and we want to thank the participants for so generously getting involved in the tasks we presented them with, and for their interest in the discussions we had. We hope that this report will be useful reading for those who were present, as a reminder of some of what we did and of what we talked about. For them, as well as for those who were not present, we have decided to present both our reflections and our activities, so that this report more truly reflects our working group.

FIRST PART : THE ACTIVITIES.

The First Day

We chose two mathematical software packages for our activities: Maple and Cabri géomètre 2. Two reasons guided our choice. The first reason is the extensive use of these packages in undergraduate mathematics. The second reason is that these are the two packages we know best, Michael being part of the team that developed Maple, and Frédéric being a keen user of Cabri both in teaching and problem solving.

A round the table introduction enabled us to witness the diversity of experience of the participants, as well as their different interests. While most participants knew reasonably well at least one of the packages, very few knew both of them well. It was also clear that very few knew Maple outside of calculus and linear algebra.

Thus we followed our plan: the next step was to work with Maple. Michael had prepared some Maple worksheets, and it is in front of computers that our work continued.

Note on the set-up of the computer laboratories

We had access to two computer laboratories. The first one was equipped with a network of 20 PCs, each with one chair (which did not encourage team work) and was used for Maple. The demonstrator (Michael) had access to a computer linked with a projector, and he used this to present the basic commands of Maple as well as the worksheets. A similar set-up was used in the second lab, with Macs instead of PCs: we used this second lab for Cabri.

As Michael was presenting the worksheets, he offered two opinions which he illustrated with some examples. The first opinion he presented is that it is okay to let the computer do some operations even if we do not know or understand how these operations are being performed, provided we can check that the result the computer gives is correct. Here are three examples which Michael presented in support of this claim.

Example 1:

Factoring a polynomial of degree 4, say, can be performed easily with Maple, and it is easy for the students to verify that the answer is correct by multiplying out the factors (again, with Maple).

Example 2:

Computing exact formulae for eigenvalues and eigenvectors of matrices other than two by two is a daunting task to do by hand but is easy for Maple to do. Again, students can easily check that a purported eigenvalue λ and corresponding eigenvector v satisfy $Av = \lambda v$.

Example 3:

The logistic equation is $y' = \rho y (Y_m - y) - h$. This can be solved exactly using separation of variables but is more an exercise in algebraic prowess than anything else. The solution can be verified simply by checking that it satisfies the differential equation with Maple.

In thinking about our discussions around this, it is interesting to note that no one asked what one would do if Maple returned the polynomial unfactored (meaning it is irreducible). How would one verify that it is irreducible? The package does not provide any additional information from which one can verify irreducibility.

There are subtleties involved in each of these examples that need to be addressed by proper recourse to theory and examples. For example, a second order linear differential equation of order *n* will have *n* solutions, each of which can be checked. But these solutions must also be linearly independent.

There is also the issue of the form of the solution that the software provides. For example, Maple insists on the form $C_1 e^{kt} \sin(wt) + C_2 e^{kt} \cos(wt)$ instead of the form $e^{kt} A \cos(wt + \circ)$.

The introduction was followed by a short demonstration of a further example taken from differential equations, which illustrates the use of qualitative and graphical methods in the study of a first order system.

The figure below is the phase portrait plot of the first order system $x'(t) = x(3 - x - y)$, $y'(t) = y(4 - 2x - y)$. The arrows show the direction field and curves are solution curves for the four initial values $(x(0), y(0)) = (0.2, 0.1), (0.2, 0.2), (0.2, 0.3), (0.2, 0.4)$. The plot depicts an unstable equilibrium at $x = 1$, $y = 2$.

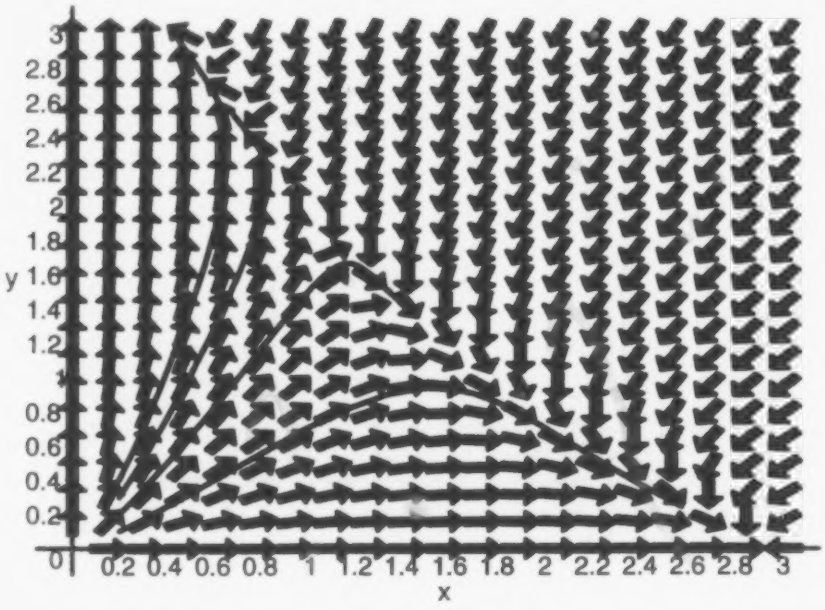

Figure 1

After this presentation, participants could look at a short Maple tutorial if they had no prior experience with Maple. Otherwise, they could look at any of the five prepared worksheets which contained assignment type questions.

1. Linear algebra: calculating eigenvalues and eigenvectors. The example illustrated what Maple does when the characteristic polynomial has irreducible factors of degree 4 or higher.

2. Differential equations: the logistic growth with harvesting model. The emphasis was on the use of graphical/qualitative methods to study the equilibria.

3. Number theory/algebra: the RSA public key encryption scheme.

4. Discrete mathematics: recurrence relations. A problem in counting the number of regions in a circle with ***n*** points on the circumference with lines between each pair of points.

5. Discrete mathematics: assymptotics. Show experimentally that the running time of two algorithms for calculating the GCD of two integers is $O(n^2)$, namely the Euclidean algorithm and binary GCD algorithm.

The work with Maple lasted until the end of our first day

The Second Day

We started our second day with a discussion following our work with Maple. We reached coffee break with an increasing need for caffeine and many questions without precise answers. Who was surprised? Certainly not the leaders of the day (Joel and Frédéric).

After coffee break, we moved to the computer laboratory to work with Cabri. Frédéric did a brief presentation of the software, and participants were given a list of problems to attack with Cabri.[1] One of these problems, drawn from an example given in John Mason's book *Mathematical Thinking,* is to use Cabri to describe the movement of the rocking horse represented in the figure below.

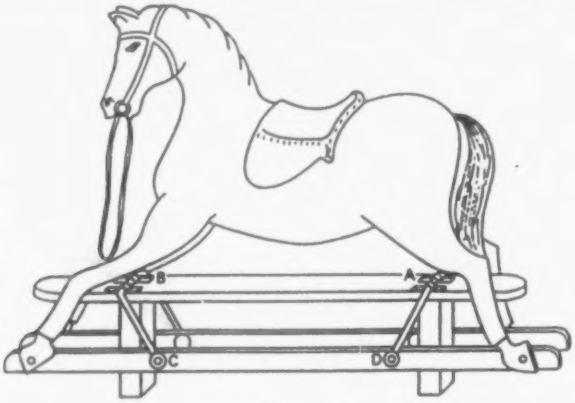

Figure 2

This problem attracted a lot of attention and caused a lot of difficulties. Some of the difficulties were due to the nature of Cabri, which is not designed for modelling physical objects or real movement, but is designed for the learning of Euclidean geometry. We will discuss below the main difficulty encountered with this problem.

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[1] Frédéric: It was challenging to teach Cabri after having discussed use of computers in the classroom the same morning. I had never taught in a laboratory setting. As I was presenting Cabri, I could notice how hard it was to teach while everyone had a personal computer they could look at and work with. My attention was often shifting from attending an individual to attending (part of) the group. Even if I tried hard to do this correctly, it certainly was very different from my usual style of teaching, and there certainly was a lot of space for improvement!

The Third day

This last day was entirely devoted to our discussions.

SECOND PART : OUR DISCUSSIONS

We will divide our presentation for this section into five parts according to the following topics: the software; programming; mathematical content and software; our teaching methods; and some concrete examples of actual use.

The software

In becoming learners again, it was both striking and amusing that the first comment made after we used Maple for awhile was: why is this so user unfriendly? Even in our group of mathematically inclined and trained people, many of us had problems using Maple. This is a fact which we can not negate nor trivialise: it is true that the complexity which is inherent to the use of Maple can be a deterrent to the use of this software.

In reaction to this complexity, we can wonder why there is not yet a system which is a lot more user friendly. One of the reasons for this is that software is not always designed with a pedagogical purpose. As one of the participants pointed out with respect to Maple:

> *The program was clearly not designed as an educational tool, and entered the education milieu at a secondary stage. Some of the advantages of the complexity in the use of Maple lie in the fact that we need to write things down: for instance, write the variables of a function. However, it would be very nice to have zooming capabilities on graphs for instance, and that is not yet included in Maple.*

Another reason is that Maple manipulates numbers and formulae as representations of functions, a task which is quite abstract compared with graphics software. Thus, there is also an inherent difficulty. One must first have a precise language for inputting formulae. Then one must have a means for directing the computer to operate on these formulae.

Notes: The current model in Maple is primarily command based: one types the commandto be executed. The palettes (for input of formulae by filling in templates) and context menus (for selecting commands to apply), new in Release 5 of Maple, are limited. Thus one is forced to input formulae and commands with the keyboard in general.[2]

Cabri offers an interesting contrast. The software was developed with a definite pedagogical purpose. The commands in Cabri are essentially the constructions which are permissible in Euclidean geometry. Thus some natural constructions which are not allowed or feasible in Euclidean geometry are not allowed in Cabri.

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[2] Michael: Even if Maple had good menus, I believe that this would not be good enough since using menus is in fact much slower than typing. The competent user soon returns to the keyboard as the primary mode of input.

As an example of this, let us go back to the rocking horse problem we introduced earlier. In modelling this, we naturally select two points A and B on two circles. It is a well-known feature of Cabri that we can then move points A and B while they remain on their respective circles.

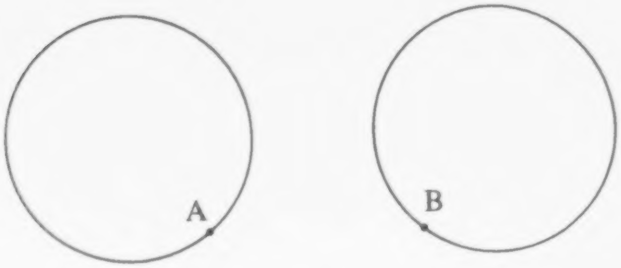

Figure 3

For the modelling of the rocking horse, the length AB is fixed. Thus we want the movements of A and B to be linked in such a way that when A moves, B will move and the length AB will remain constant. To achieve this, one cannot simply fix the length after choosing A and B. The Euclidean way to achieve this result is to choose A first (on one circle), and then choose the given length L. The point B is then constructed as one of the points of intersection of a circle of radius L centred in A with the original circle.

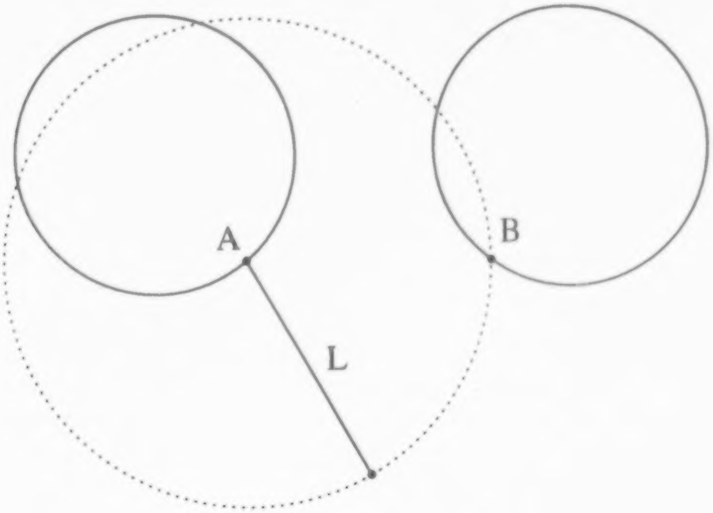

Figure 4

Such a construction is natural in Euclidean geometry, but is not necessarily natural outside of this framework. Is the solution to use Cabri uniquely in the pedagogical context for which it was designed?

And, in general, should it be that software be designed with a precise pedagogical purpose and then be used only in a specific context?

While this may appear to be a good idea, some participants question this. Using a program in a precise and well-defined pedagogical setting is yet another constraint: the role of the teacher should not be that of a technician implementing a structured and linear set of activities. Instead, shouldn't we accept the inherent complexity of a general purpose package and adapt its use to our needs rather than to try to have a program that is beautifully tailored for a precise, but very limited, usage? This question was not resolved during our working group, and remains a topic open to more discussion.

However, this question led us naturally to discuss the usage of Maple, and the adaptation of Maple to course work. Worksheets are the standard way of doing this, and are certainly the most common way of illustrating the utility of Maple. What about programming in this respect?

Programming

There has been a shift in most mathematics curriculum away from programming. How is this related to the programming needed for Maple? Do we need to learn more than the programming needed at the worksheet level in order to be able to properly use this software?

Considering a worksheet as a type of program, the question is: do students need to learn more programming than what is needed at the worksheet level? The experience of some participants is that many students are little inclined to programming, and that they must do more programming than what is strictly necessary in order to overcome the barriers they have concerning the use of Maple. Others think that learning programming with Maple, and learning only what is necessary to program at the worksheet level, is sufficient.

It may be important here to distinguish between two types of programming. Doing interactive calculations which involve simple commands and loops may be considered different from writing subroutines—perhaps only the later is really programming. (See Monaghan 1997 for actual examples of algorithms taught using Maple.)

In our discussions, all seemed to agree that students should not only be users of worksheets: they should also have to create their own worksheets (in some courses). Practically, this work will take time away from some of the other type of work to be done: using a software is not without an impact on the content that we have time to cover, and it does not only save time! Thus using Maple actively (and not solely for demonstrations) implies selecting course content, and not doing everything we used to do.

What about Cabri and programming? It was interesting that some experienced Cabri users did not feel that they were programming when working with Cabri[3], unless they were constructing macros. Even then, this hardly seemed to them to deserve to be called real programming! In reaction to this impression, most participants seemed surprised. Obviously, working with Cabri was, for most participants, programming, although in a format which is very different from traditional programming; what used to be called programming was not an activity performed in an interactive mode in such an easy and exploratory way.

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[3] Frédéric: I am one of those experienced users who did not associate Cabri with programming. The conversation made me change this conception, but leaves me wondering about the different types of programming.

Mathematical content

Inevitably, we had to talk about the differences between mathematics, calculations, concepts, understanding, proving, etc.

The first aspect we discussed was the distinction between mathematics and computational mathematics. Is there really such a thing as mathematics as opposed to computational mathematics? While it is clear that our training and early mathematical experiences have tainted our vision of mathematics, it is also clear that different experiences will yield to different ideas of mathematics. Thus there is a need to reflect on our past education when discussing this issue, and a need to keep in mind the fact that current education will affect the perception of mathematics by those who are currently our students.

The point is that while nobody said that calculations were the whole of mathematics, nobody in our group said that calculations are not mathematics. Of key importance to many is the link that can be made between the calculations and the concepts that lie behind those calculations. For instance, to be able to see differential equations geometrically can lead to a different and better understanding of some aspects of differential equations.

Venturing a little more on differential equations, it seems that for most (and perhaps all) participants, differential equations theory as learned in the undergraduate curriculum was a series of recipes for solving classes of differential equations, based only on algebraic representation. That the geometric representation can lead to a better understanding in some cases seems clear. Furthermore, some suggest that spending time learning how to effectively solve by hand many different types of differential equation is not what we should do. Instead, learning one basic technique and leaving the rest of the work (including some standard tricks as well as some more sophisticated work) to the software is seen as a more appropriate way of teaching the subject (given that one can check the solutions obtained by the software are indeed correct).[4] This view is compatible with the time constraints that the extensive use of the software imposes: there is less standard material to cover, which leaves time for working with the software.

A similar point is made for eigenvalues, their utility being better understood when we have time to work with eigenvalues and eigenvectors instead of spending time calculating them by hand (when this is feasible). Of course, a good geometrical understanding of 2 by 2 matrices can lead to much understanding, and this is not where the simplification of calculation is of such great importance. (More examples where simplifying the calculations can really help understanding are given by solving linear systems, inverting matrices, etc.)

Another aspect of calculations is that when calculations are being performed by a program, the user may not know how the calculations are being done. Once again, we can take factoring a degree 4 polynomial as an example. Most users do not know what algorithm is being used to perform this operation, and are then using a black box. Indeed, none of the participants knew any algorithm for factoring polynomials of degree 4 or higher, but we all seemed comfortable using the factor command!

In some cases, one could argue that the calculations should be taught first. But this is hardly realistic in complex instances; the algorithms used are simply too complicated. Thus one should not (and perhaps cannot) always unravel the mysteries of the black box. This leads us to accept some results that we do not actually know how to obtain ourselves.

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[4] Michael: We can also note that the same comment applies to learning techniques of integration.

However, Cabri is also different in this respect because the basic operations done with Cabri are basic geometrical constructions, and are not (in that sense) black boxes. Also, in some way, we cannot say that we have checked that a result is true with Cabri, but rather that we see that the result is true. It is often argued that the strong visual evidence provided by a dynamic geometry software leads us to wonder why a result is true: it certainly works that way for us! Thus it appears reasonable that the same should be true for our students. While some research done in France supports this idea, can we apply the same conclusions here? The mathematical curriculum and the culture in France being so different from ours, it is not clear that our usage of a dynamic geometry software will produce the same results.

Furthermore, there is some confusion for students as to what we should accept (for instance, that the bisector is the bisector when we construct it) and what we should prove. Even if such a distinction is clear for us, we cannot use the software without paying attention to the new elements being introduced by this use, and in particular the introduction of statements by the software, some of which should be accepted as such while others should be proved.

It is also important to note, when comparing the two programs, that while Euclidean geometry has an important visual component, manipulating matrices of formulae may not. Verifying that things are going right in Cabri is made easier by the visual picture: a wrong instruction will lead to a picture which is not the one intended. However, we do not have such visual clues when we are working with formulae.

Our teaching methods

While it is still true that the only guaranteed contact of students with the software is often the demonstration in class, the teaching style and the teaching methods are dramatically affected when we can teach in a setting where students have access to computers in the classroom.

The mere fact that the student has a computer screen to look at changes the student-teacher dynamic. Adding the fact that the student will work with the computer creates new situations. For instance, a question asked by a student can refer to something which he is the only one experiencing (for instance, a problem with a wrong command), and the teacher attending that individual need cannot always relate it to others in the class. This is not the same in traditional teaching, where most questions asked by students are understood by all, and where the answer provided by the teacher can be of interest to many students.

How can we move, as teachers, beyond this? Most of us did not have any role models for teaching with computers, and we need to get some experience, readjust our timing, and reassess our teaching style and objectives in order to effectively use software in our classes. Unfortunately, we did not have time to discuss this question at length, and thus there is nothing more to report on this subject.

Some examples of actual use

Answering a question about the issue of assessment, a few examples were offered. The context is as follows. While it seems obvious that in order to effectively use any software in a mathematics course, there needs to be a significant part of the assessment which is actually based on work with the software, to do so is often difficult.

Some have experienced evaluations in a class equipped with enough workstations for each student to have access to one. In this setting, giving enough time and a suitable task can be a mode of evaluation.

We did not exchange much about this option, as it appeared that most participants did not have access to such facilities, often because of the group size.

What can we do then? One of the issues is to find good examples, assignments or projects: these must be problems where the computer will really help. It is much easier to find problems for traditional mathematics as numerous books are available, but there are not as many resources for problems to be done with a real help from the computer. (For some concrete examples, see Guidera and Monagan 1997)

Michael explained how he had set a significant project in one of his courses (discrete mathematics). The project lasted 2 months, and was done in pairs. He graded the 125 projects, which each comprised an exploratory and an written part. The trade off was that he had to cut one week out of 13 weeks of material in order to do this. However, the result was that some students reported that the project was the most valuable part of the course. Michael felt that the material covered by the project would stick, and that the trade-off was definitely worth it.

In reflecting about this, the following question was posed: is Maple becoming content? Is a commercial product becoming a content that we teach? How do we clarify the role and status of such a software, which is aimed to become a system that does everything?

One possible way to analyse this is by looking at four aspects of mathematics: visualising; modelling; ways of thinking; proving. The impact of Maple on each of these aspects can be different. For instance, it clearly has an impact on our way of thinking in terms of graphical, numerical and symbolic aspects. This change in our ways of thinking is an important one to take into account in the content and in the curriculum as a whole.

Moving from this example, we discussed about the current use of mathematical software. How are they effectively being used in our institutions? How are they used for concept building? How is their use changing the traditional style of teaching? How has it changed, in reality, the content of the courses?

Participants offered some good examples, but it was clear that, in many institutions, the students themselves do little work with Maple and other software (with some notable exceptions). In general there are some in class demonstrations and the use of graphical capabilities in relation to local linearity, partial derivatives for 3D plots, and the understanding of bifurcation points (prior to the formal mathematical treatment). Also some use in linear algebra is explained: for instance, witnessing the actual matrix in Cayley-Hamilton rather than just seeing a 0 and other symbols and losing track of what they actually mean.

Michael commented, after the workshop, that Maple's impact on proving is clearly lower than its impact on other aspects of mathematics. However, one surprising place where it can be used to automate proofs is in algebraic proofs by induction of identities of the form $^{n}\Sigma_{i=0}\ f(i) = g(n)$, as can be seen with the following common question. Prove the identity $^{n}\Sigma_{i=0}\ i^2 = 1/6\, n\ (2n + 1)(n + 1)$ by induction.

Assuming that the identity holds for $n = k\text{-}1$, to show that it holds for $n = k$ requires that we show that $1/6\,(n - 1)(2n - 1)n + n^2 = 1/6\, n\,(2n + 1)(n + 1)$. What usually happens next is an attempt to convert the left-hand-side of the identity into the right-hand-side by mathematical wizardry. Instead, subtracting the right-hand-side from the left and trying to show $1/6\,(n - 1)(2n - 1)n + n^2 - 1/6\, n\,(2n + 1)(n + 1) = 0$. is simpler for computers, and simpler for us too! The simplify command will readily output that this is indeed zero.[5]

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[5] Michael: Clearly in this example we are suggesting that the computer be used to assist in the proof by induction, not to *do* the proof by induction. However, proving summation identities by induction could be

CONCLUSION

After this working group had completed its work, Joel Hillel sent a few comments in relation to the use of CAS (Computer and Software), which seemed to us an appropriate way to conclude our report.

Discussing the potential benefits of CAS in undergraduate teaching makes more sense when grounded in a specific context. At the very least, we should articulate some answers to the following questions:

For whom?

For what purpose?

How?

For whom?

Who are the targetted students: Mathematics majors? Prospective teachers? Engineering students? Are they students with relatively strong background? Do they take maths because they like the subject or because they have to do some math courses in order to get into other programmes?

For what purposes?

Just looking at the use of, say, Maple, in a linear algebra course, there are different scenarios. Each scenario contains some implicit assumptions about the background knowledge possessed by the students and the way CAS activity is integrated with lectures. The scenarios include:

- Computing
  Example: find the inverse of a large matrix
- Verifying general results in particular cases
  Example: finding the inverse of a matrix A using Gauss-Jordan elimination and verifying that the matrix obtained is, in fact, an inverse.
- Providing inductive evidence for theorems
  Example: check that $f(A) = 0$ where $f(x)$ is the characteristic polynomial of A.
- Becoming familiar with objects and operators
  Example: investigating properties of Vandermonde matrix
- Dealing with common students' misconceptions
  Example: are n vectors chosen randomly (in a suitably defined way) in $\Re^2$ more likely to be dependent or independent?

How?

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done entirely by the computer.

How exactly is a CAS integrated into a course? How much time are students expected to spend in the lab? How does the lab count? As an official class activity, instead or in addition to lectures; as optional; with or without resource persons in the lab; with explicit computer activities (worksheets for instance) or as a loosely defined investigative work?

Final words

While the questions above are not complete, they may serve as a background to help describe activities with computers and software, and they outline the diversity of the situations and the possible use of CAS. In this way, they may be a good way to end our formal report.

We (Frédéric and Michael) want to thank the organisers of the meeting for providing us with the opportunity to lead this working group, and we wish once again to thank all the participants for their willingness to participate fully in our working group.

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TOPIC SESSIONS

PROFESSIONAL DEVELOPMENT
FOR PRESERVICE MATHEMATICS TEACHERS

Nadine Bednarz, Université du Québec à Montréal
Linda Gattuso, Université du Québec à Montréal

INTRODUCTION

Recent changes in Quebec primary and high school curricula (Ministère de l'Éducation du Québec, 1997) and in teacher training (MEQ, 1992) emphasized professionalism in teaching and teacher interventions. Being a mathematics teacher and acting as a professional means developing abilities to understand student productions (their reasoning, their errors, their conceptions), developing new ways of doing things, of understanding the reality of learning and teaching mathematics. For several years, our involvement in teacher training (Bednarz, in press-a; Bednarz, Gattuso and Mary, 1996; Bednarz, Gattuso and Mary, 1995) and our research on questions related to the professional development of teachers (Bednarz, in-press-b; Bednarz, 1998), have shed some light on components of professionalism in teaching. However, some specific questions related to teacher training still have to be answered:

- How can we intervene as teacher trainers so that students construct for themselves this teaching knowledge?

> We know more about the way students learn mathematics at school than we know about the way preservice teachers learn to teach mathematics. (Bauersfeld, 1994)

- How can we prepare students to be mathematics teachers and to act as professionals, in other words, to develop new ways of seeing the learning and teaching of mathematics?

It is well known that student teachers entering university bring with them their previous conceptions of mathematics, its learning and its teaching. Studies show that these conceptions, built up over 12 or 13 years of prior schooling, reveal a mechanistic view of mathematics and of the way it ought to be taught (Bednarz, Gattuso and Mary, 1996; Kagan, 1992). These studies also reveal that in their own classroom practice, the student teachers adhere to their previous views, and tend, when facing a problematic situation, to resort very quickly to a certain *habitus* (Bourdieu, 1980).

> Through these typical regressions, the functioning of this kind of *habitus* readily supports the "old" solutions and the reproduction of the old school. (Bauersfeld, 1994, p.179)

- Consequently, what are the conditions that can affect the teaching practice of preservice teachers and their ways of interpreting student productions?

The teacher training curriculum cannot ignore the student teachers' prior experiences and must efficiently counterbalance their previously formed views of mathematics and teaching.

We will present various interventions aimed at having preservice teachers reach this developmental goal over the 4 years of training. The interventions proposed are founded on our conception of professional development for mathematics teaching. First, we will make a few remarks regarding the origin and the development of our program.

1. PRELIMINARY REMARKS

The aim of our high school mathematics teacher training program was to contribute to the necessary changes in student teachers' views of mathematics teaching which are fundamental to their future practice. The interventions presented in this paper are the result of a gradual restructuring of the program by a team of mathematics educators (*didacticiens*) over a period of more than 20 years. The team consists of about 14 professors of the mathematics department of UQAM (Université du Québec à Montréal), working in teacher training since 1970.

This teacher training program has been designed in terms of a 4-year coherent curriculum not only in terms of local interventions during courses. This program was not created from an *a priori* theoretical framework. On the contrary, the model is the result of a reflection on our action as trainers that led to an *a posteriori* analysis. The underlying model developed "in action" provided a framework for the representation of a professional development of teachers.

One important point that permitted the development of this curriculum is the fact that the didacticians (in the French sense) were involved in the teaching practicums as supervisors since the beginning of the training program. This fact contributed to bringing into our own university courses, a variety of observations of pupils in classrooms (their reasoning, their errors, their difficulties, etc.) and also observations of the teaching of different subjects (problem solving, teaching of rational numbers, of algebra, geometry, etc.). This collection of observations served as a basis for reflecting on the learning and teaching of mathematics.

Our classroom presence also contributes to linking the teacher training program to the reality of schools. Thus the program is oriented towards intervention in classrooms and reflection on this action. The student teacher, for example, has, at different moments of his training, to prepare lessons, to test them and to analyze them . Finally, this involvement on the part of the didacticians in teaching practice also favored the creation of solid collaboration with teachers in high schools. Some of them, for example, work with the didacticians in the courses at the university and in the teaching practicums.

The interventions developed by the team are also supported by their own research on learning and teaching mathematics. These studies not only bring a better understanding of the reasoning, errors, difficulties of students in different areas but also aid in the design of teaching experiments.

The following table (Table 1), presenting schematically the teacher training curriculum, gives a global idea of the importance of the involvement of the team of didacticians. The shaded areas are courses and teaching practicums for which the team of didacticians is responsible.

The courses are not only mathematics teaching courses (*didactique des mathématiques*) but they also cover mathematics, history of mathematics, and activities related to computing and teaching of computing (Table 1). The mathematics courses given by the didacticians are structured with the preoccupation of initiating students into an alternative culture of mathematics (Bednarz, Gattuso and Mary, 1995). In the following section, we will concentrate on the mathematics teaching courses (*didactique des mathématiques*). Overall it is important to underline that the curriculum is more than a juxtaposition of different courses. A thread links the different training activities during the four years of the program. We will present the principles underlying the teacher trainers' interventions.[1]

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[1] In the present context (UQAM), the teacher trainers and the didacticians of mathematics are the same persons.

Table 1-Baccalauréat en enseignement secondaire: concentration Mathématiques

Options : Informatique , Physique et Initiation à la technologie[1]

1^re^ année		2^e^ année		3^e^ année		4^e^ année	
A+0 Ses.I	H+1 Ses.II	A+1 Ses.III	H+2 Ses.IV	A+2 Ses.V	H+3 Ses.VI	A+3 Ses.VII	H+4 Se.VIII
TC1 3cr ÉDU 1000 Introduction et initiation à l'intervention pédagogique	TC2 3cr PSY 2010 L'adolescence	TC4 3cr ÉDU 2102 Relation éducative et gestion des situations d'apprentissage (intensif)	TC6 3cr ÉDU 5010 Évaluation des apprentissages au secondaire	3cr MAT 3225 Didactique de la variable et des fonctions	TC13 3cr Culture générale *(intensif)*	TC9 2cr ÉDU 6002 Problématiques interculturelles à l'école secondaire *(intensif)*	TC10 3cr ÉDU 6005 Éducation, épistémologie et méta-cognition
3cr MAT 2700 Structures numériques	TC3 2cr ÉDU 2005 Fondements de l'éducation et enseignement au sec.	2cr MAT2026 Didactique de l'algèbre *(intensif)*	3cr MAT3035 Géométrie II	3cr MAT 3106 Initiation à analyse	3cr MAT3224 Didactique des mathématiques II *(intensif)*	TC5 3cr ÉDU 3060 L'adolescent en difficulté d'adaptation et d'appren- [illegible]	TC8 3cr HIS -SOC- PH Formation fondamentale
3cr MAT 1030 Géométrie I	3cr MAT 2024 Didactique des mathématiques et laboratoire	3cr MAT 2225 Raisonnement proportionnel et concepts associés *(intensif)*	3cr DIN 1200 Progic. PHY1050 Mécanique PHY2000 Mécanique	3cr DIN 4200 Progr. PHY 1770 Ondes EDU2080 Démarche techn.	3cr MAT 1085 Probalités et statistiques *(intensif)*	3 cr MAT 6221 Histoire des mathématiques *(Intensif)*	TC11 2cr ÉDU 6025 Profession enseignante et milieu scolaire
3cr DIN 2200 Programmation dans l'enseignement			3cr DIN 3200 Multim. PHY 2270 Élect. PHY 2005 Élect.	3cr DIN 3300 Inforoute PHY 4500 Thermod. DIN 3300 Inf			2cr ESM 6220 Séminaire de synthèse
4cr [illegible]	3cr MAT 2812 Applications pédagogiques de l'informatique dans l'enseignement des mat.	STAGE2 (mathématiques) ESM 3202 Initiation à la pratique de l'enseignement (24 jours) ESM 3201 1er Séminaire: d'intégration et d'intervention	3cr DIN 5300 Hist. Optique EDU 2085 Archit.	3cr DIN 5200 Di info Simulat. MIC 3000 Syst. inf	3cr DIN 6200 Proj. ESM 3511 Did. ESM3515 Did. *(intensif)*		DIN [illegible] ESM 3505 Hist. ESM 3506 Hist.
	STAGE1 2cr ESM 2000 Immersion en milieuscolaire *(8 jours)(non disciplinaire)*				STAGE3 (opt) 7cr ESM1200 Inf /ESM 4502 /Init.T/ ESM4501/ Phys. *(20 jours)*	STAGE4 7cr (mathématiques) ESM 6201	3cr Choix info. PHY5260 Phys. mod. PHY 2020 Matériaux
16 crédits	16 crédits	12 crédits	15 crédits	15 crédits	15 crédits	15 crédits	16 crédits

TCX:	Cours du tronc commun en psychopédagogie	=>	30 crédits
MATH:	Cours disciplinaires et de didactique	=>	47 crédits
INF. PHY., INIT. TECH.	Option	=>	27 crédits
STAGEX	Stages en milieu scolaire	=>	16 crédits
Total 120 crédits			Juin 1996

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2. UNDERLYING PRINCIPLES

The courses are centered on the development of professional abilities and this is done across many courses.

-We firmly believe that to develop professional abilities, the student teachers have *to learn to observe pupils' strategies, to anticipate their difficulties, their errors, their reasoning, to learn to question them adequately, to diagnose errors, to take into account students' questions and productions,* etc.

In order to develop these abilities, a repertory of pupil errors gathered by the trainers through observations of teaching practice allows the student teachers to analyze real pupil solutions. Videos showing pupils solving problems, or discussing during classroom situations, also provide some working material. The student teachers are also encouraged to interview high school pupils and explore for themselves situations previously discussed in class. Again, the results of their interviews are opportunities for discussions about errors, conceptions developed by pupils, ways of presenting situations, etc.

As we can see, analysis of pupils' productions is at the core of the activities of student teachers. In the same way, the didacticians use their students' productions as working material for their classes.

-Secondly, the student teachers also have to develop competencies while experimenting the teaching cycle: lesson planning, experimentation and analysis.

In the first mathematics education course (*Didactique des mathématiques I*), preservice teachers have to prepare a set of three lessons and present one in front of their peers who give their reactions. Finishing students (in the last year of training) act as counselors for the preparation of the first lesson presented in front of their peers. They attend the presentation of the lesson and afterwards they write a critical analysis of the experimentation and the suggestions they made. A high school teacher also attends the presentation of lessons and gives feedback. The student teachers will then have a chance to observe their lesson on video, to consider the comments of both the high school teacher and their peers and to prepare a second version and present it again. This short term planning will be experienced more thoroughly during the first teaching practicum, where the preservice teacher teaches two groups of students for about two weeks.

In a subsequent course, student teachers work on long term planning. To achieve this planning, they must not only develop a conceptual analysis of the subject to be taught, but also prepare a set of problems and anticipate the different solutions which could be used by students in order to situate where the students are at. They also have to find out what problems the pupils should be able to solve at the end of the lesson sequence (kind of problems, reasoning, procedures....) and present the teaching approach they will choose to reach their goals. The result is what we call a *teaching sequence.*

They will again prepare such a teaching sequence for their last teaching practicum, where they will take charge of the classes of their host-teacher for eight weeks. In their teaching practicum report, they will present this teaching sequence with the changes and modifications they made or would make.

In all cases (different courses and practicums, at different moments of the training program), student teachers are confronted with the three stages of the teaching cycle: planning, teaching and reflecting.

Certainly, the most important element in this work is what we call the *conceptual analysis*. It consists of an analysis of the concept from three closely linked points of views. First, it is important to look at the concept from a mathematical perspective: its characteristics, its properties, the fundamental reasoning involved, the related concepts, the prerequisite concepts, the future use of the concept, etc. The second point of view is that of the learner: what are the misconceptions, the epistemological obstacles encountered

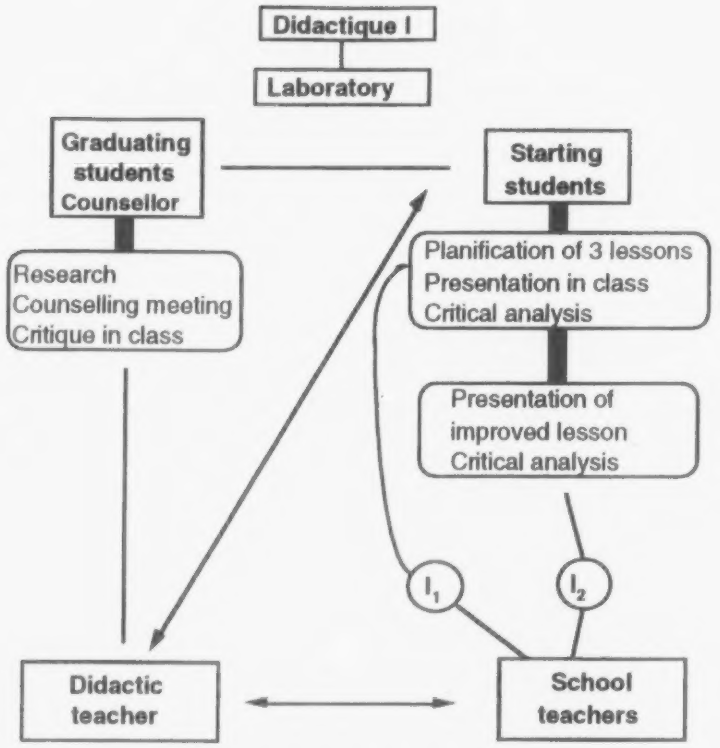
Didactique I
Laboratory
Graduating students Counsellor
Starting students
Research
Counselling meeting
Critique in class
Planification of 3 lessons
Presentation in class
Critical analysis
Presentation of improved lesson
Critical analysis
I1
I2
Didactic teacher
School teachers

Figure 1: Organizational schema of the course *Didactique 1*

by pupils, the difficulties and reasonings they use? Then there is the teaching point of view: what activities, material, problems should be planned for the learning of this concept; what didactic variables would engage pupils in one kind of reasoning more than another? It is important to underline that the three points of view are closely interrelated and cannot be examined separately.

- A third principle underlying the different courses is that student teachers must *develop the ability to analyze mathematical activities.*

A teacher has to be able to prepare an adequate learning situation. Not only must the situation be mathematically correct but it has to respond to the needs foreseen in the conceptual analysis (difficulties, misconceptions, etc.) and favor the construction of meaning by pupils. A teacher then has to be able to analyze different problems and contexts, to write some new ones to introduce a concept involving didactic variables (numbers, context, organization of the activity in classroom, formulation, etc.) with the aim of improving pupils' strategies.

- Finally, in the different courses there is a preoccupation to *develop mathematical teaching abilities: verbalization, representation, contextualisation, adequate use of teaching material.*

Student teachers will on a regular basis have to verbalize mathematics, using different levels of language, so that pupils can understand the underlying meaning (for example verbalization of fraction, of equivalence of fractions in a context with the different meanings of fraction—part-whole, division, or rate;

verbalization of an equation, etc.). In all cases—using verbalization, contextualisation, representation, or materials—the concern is to build meaning for concepts and symbols. Preservice teachers will have to encourage their students to verbalize their reasoning and their symbolic work. To do so it is very important for the student teacher to realize that, he himself/she herself has to be able to verbalize adequately.

These underlying principles are already present in the first course of mathematics teaching at the beginning of the 4-year curriculum, as can be seen in the course objectives.

Didactique 1 Course Objectives:

A. Student teachers must acquire mastery of the content and competencies in didactic analyses.

When a problem, an activity, or an exercise is proposed, student teachers must be able:

- to carry it out as an expert (explain the reasoning and verbalize it, ...),
- to identify the procedures that the pupils would use,
- to pinpoint the abilities and knowledge that the pupils will use to solve this situation,
- to underline the principal didactic variables, i.e., the elements of the situation (instructions, numerical values, nature of the objects…) the teacher can vary to provoke changes in the pupil procedures,
- to predict the principal errors and difficulties.

B. Student teachers must also be able to analyze students' productions.

Analysis of students' productions is done with the hypotheses that there is a logic behind the students' productions. While analyzing, one must be able to describe the procedure produced by the pupil and to find the origin of this procedure. It is necessary to formulate some hypothesis on how the pupil conceives the task, on how he reads the context, on the influence of the choice of didactic variables, on the pupil's concepts....

C. Finally, student teachers must be able to structure classroom activities (outlines and detailed lessons).

- They must identify the aims of the activity, describe the pupils.
- They will describe the lesson in detail: the choice of the didactic variables, the organization of the class, the different stages of the activities, the role of materials.
- They will make an *a priori* analysis: imagine the procedures that could be used by pupils, their difficulties, their questions, their reactions.

These professional abilities are developed through different interventions done during their courses. We will present a few of them.

3. EXAMPLES OF INTERVENTIONS

The concerns of teachers in classrooms are always present in the different courses, particularly with respect to the following aspects:

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3.1. The pupils, their difficulties, their reasoning, their conceptions

Preservice teachers are encouraged to interview high school pupils on different subjects. In the next figure (Figure 2) you see an example of questions used in an interview done by student teachers.

The following instructions are given to the student teachers:

"We asked Frank the following question: find the value of $x + z + 9$ if $x = 5$ and $z = 5$. Frank answers that it is impossible. Is he right? "

"We asked Sylvie the following question: simplify if possible $3y + 8x + 2y$. Sylvie answered $13xy$. Is she right? If she is wrong, correct her answer."

"What is the area of the rectangle?"

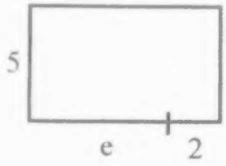

Figure 2

- Ask these same questions to the pupils.
- Note the questions of the pupils.
- Note their hesitations, their attitudes, etc....
- Keep their written work.
- Reproduce the pupils' solutions on overhead transparencies.

Here is an example of the answers brought back by a student-teacher (Figure 3)

The student teacher notes the pupil's answer, the pupil's comments and the beginning of his analysis and presents it to the class. And again, as we didacticians ask student teachers to observe the pupils' solutions, in the same way, we observe our students' productions and work from them in our course.

3.2. Diagnosis of Errors

The analysis of errors has another aim as well: to prepare student teachers to diagnose errors. Starting from the repertory of real pupils' solutions gathered by the team of didacticians while supervising teaching practice, student teachers will have to identify errors, compose problems or situations which will permit them to confirm their diagnosis (situations where the same erroneous reasoning produces a good answer, and situations where this same reasoning produces an error). The following example (Figure 4) illustrates this activity.

The student teachers work on the following tasks:

First answer the question yourself.
Describe the pupils' errors.
Identify the circumstances where the error appears.
Compose a similar problem where the reasoning used produces the same error.

Compose a similar problem where the reasoning used produces a correct answer.
What immediate action can you suggest that will lead the pupils to question their error?

MÉLANIE 14 years old
third year high school

SITUATION PRESENTED TO MÉLANIE
"We asked Frank the following question:
Find the value of $x + z + 9$ if $x = 5$ and $z = 5$.
Frank answers that it is impossible.
Is he right?"

MÉLANIE'S ANSWER:
Yes, the teacher told us that the letters had no value.

COMMENTS NOTED BY THE STUDENT TEACHER:
(Mélanie says she...)
Did not learn the value of the letters.
Does not understand why letters have a value.
Says: "If you pay no attention, you will add the letters, but you shouldn't, you leave them as they are. My teacher says you shouldn't add letters."
With what she knows she would say that it is impossible.

ANDRÉ'S ANSWER:_
The letters are variables.

COMMENTS NOTED BY THE STUDENT TEACHER:
Afterwards, for this same equation, André gave different values to x and z. One example he gave was: $x = 5$ and z, no value.
After some explanations, he was able to do every example.

Figure 3

You are the teacher in first year of high school. You give the following exercise to your students.

Write the following numbers in increasing order
2.46 2.54 2.3 2.052 2.32

Many pupils give this answer:
2.05 22.3 2.32 2.46 2.254

Others write:
2.052 2.254 2.32 2.46 2.3

Figure 4

3.3 Situation Analysis

Student teachers have to work on situations and make choices. For example, working on problem solving, student teachers will analyze different problems, their difficulties, the errors they can provoke and compose new problems with contexts, involving a particular operation, following certain constraints. This reflection gives them a better understanding of the complexity of these different problems in arithmetic and allows them to perceive the transition from operations on natural numbers to operations on rational numbers (meaning of the operations).

In the following figure, an example of a test question, illustrates what student teachers are supposed to be able to do at the end of the first course in didactics.

Transition from N to Q

The following numbers have been chosen to work on division, extending it from N to Q.

$8 \div 2$ $\quad 8 \div 2/5$ $\quad 8 \div 10$

$8 \div 2/3$ $\quad 8 \div 6/5$

a) Compose a division problem with the meaning "grouping" and prepare a visual representation and an appropriate verbalization to cover every case (apply it to the case $8 \div 2$).

b) Say in what order you place the different cases so that the graduation of problems can be the most appropriate, underlying the moments where there is a new difficulty or a conception to take account of.

c) Show, with the help of the visual representation and the verbalization, how we can prepare the answer for $8 \div 2/3$ and $8 \div 6/5$.

Figure 5

3.4. Contextualisation, Representation, Verbalization and Material

On many occasions during their courses, student teachers will have to verbalize their reasoning, to find contexts or representations that could be useful to introduce concepts or to solve a problem, and to reflect on the role played by teaching material in the learning of mathematics. To help student teachers realize the importance of these tasks, we ask them to solve a division problem in two different contexts (a first one, with division as sharing, and another with division as grouping). This has to be done with the added constraint of using concrete material, drawings or symbolic notation.

Instructions:

Divide the class in 5 teams.

a) Write a problem requiring a division with the sense of "sharing" and another with the sense of "grouping" using 258 ÷ 6.

b) Do the division using the suggested material and verbalizing the reasoning in the chosen context for each of the problems you wrote.

Team 1)	Multibase blocks
Team 2)	Abacus
Team 3)	Colored buttons
Team 4)	Drawing of multibase blocks (on overhead transparency)
Team 5)	Symbols (numbers)

Using their material each team will present their solution orally in front of the classroom.

Figure 6

Student teachers can then observe that the manipulation of material and the verbalization differs with the meaning chosen for the operation. Some verbalization is more complex than others. They also differ with the material or the mode of representation used. It is an occasion for a reflection on the possibilities and limits of the different teaching material used.

3.5. Enhance Links Between Concepts

During the elaboration of lessons, or the elaboration of the teaching sequence, preservice teachers have to examine the learning of mathematics from a longitudinal point of view. They have to analyze links between concepts (the learning of one concept is not independent of the learning of others). For example, when they are working on the teaching of fractions in first year of high school, it is important to look at what has been done in elementary school on the same subject: what are pupils supposed to be able to do? How can we go further (transition to rational numbers)? Also, how can we prepare the learning of future concepts (for example in algebra, solving equations with fractions, etc.).

We also encourage links between reasonings, for example, the development of certain reasonings in mental computation. While counting mentally, we will say: "to find 93 - 27, we can find 93 - 30 = 63 and then readjust, 63 + 3 = 66." We simulate, find something suitable and then readjust. This mental activity prepares certain reasonings in algebra. For example, the completion of the square $x^2 + 3x + 7 \ldots \longrightarrow x^2 + 3x + 9/4 + 7 - 9/4$, is similar to finding something suitable and readjusting.

4. THE UNDERLYING DIDACTICS OF THE TRAINERS

If we want preservice teachers to change their usual habits in mathematics teaching, the training approach has to be coherent with our aims. From that perspective, didactics should not be taught as a science (what we know about learning and teaching of mathematics, the results of various studies in different areas, the didactic theories, ...) and our courses should not be conferences on what to do in classrooms. It is more a training by didactics that is proposed here (in the sense that didactics constitutes a framework for the trainers to choose the situations proposed to students, and the way they are used). As we have seen previously, with their conceptions and reasonings in mind, we place the student teachers in different classroom situations which they have to experiment and reflect on. Our aim is to provoke an evolution of their views about mathematics learning and teaching. Questioning, explanation of different points of view, interactions between students and teachers play an important role here. By doing so we want preservice teachers to develop abilities to make decisions, to make appropriate choices, to organize themselves, to question their pupils, etc. In this way, we hope that preservice teachers will construct for themselves a repertory of meaningful knowledge and pedagogical strategies about the teaching and the learning of mathematics.

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Topic Session 2

HOW DOES THE ACHIEVEMENT OF CANADIAN STUDENTS COMPARE TO THAT OF STUDENTS IN OTHER COUNTRIES?

David F. Robitaille
University of British Columbia, Vancouver

The achievement of Canadian students in mathematics and science compares very favorably to that of students in most other countries, including those of our most important trading partners. Results from the Third International Mathematics and Science Study (TIMSS) indicate that the performance of elementary and secondary school students in this country in mathematics and science is significantly higher than the international average on almost every one of the major comparisons made as part of the study. The results also show that there is considerable variability in performance among provinces, with Alberta being consistently strong.

Over fifty countries participated in one or more aspects of TIMSS, and the results of the achievement testing components of the study have been released in four stages over the past eighteen months. TIMSS, the largest study of its kind ever conducted in the field of education, is a project of the International Association for the Evaluation of Educational Achievement (IEA).

NATIONAL AND INTERNATIONAL COMPARISONS IN EDUCATION

Over the past decade, interest in international comparisons of educational systems has grown dramatically. The number of countries that decided to participate in TIMSS is at least twice as large as the number of countries that has participated in any previous international comparative study, whether that study was sponsored by IEA or any other organization.

What accounts for this level of interest in international comparisons? Many, perhaps all of the participating countries, are interested in comparing their educational programs, instructional practices, and student outcomes with those of other countries (see Hussein, 1992, for example), especially those that are important to them for political, cultural, linguistic, or economic reasons. Also, there is a widely held view that a nation's continued economic well-being and its ability to compete in the global marketplace are strongly linked to how well that country's elementary and secondary school students do in school, and especially in mathematics and science. Whether one supports the notion of such a linkage or not, the underlying belief is one that motivates many individuals, institutions, and governments to look to studies such as TIMSS for relevant information.

TIMSS was designed to respond to the disparate reasons motivating countries to take part in large-scale comparative studies and, in so doing, to adhere to the highest standards of quality for conducting research into educational practices and outcomes. The overall goal of the study was to compare the teaching and learning of mathematics and science in elementary and secondary schools around the world, in order that participating countries might learn from one another about exemplary practices.

The conceptual framework for TIMSS is based on the centrality of curriculum to an understanding of what goes on in classrooms and what students learn: the curriculum as intended at the provincial or national level, the curriculum as implemented in classrooms, and the curriculum as attained by students number of educational inputs and processes that previous research has shown to be linked to differences in outputs, including students' attitudes and achievement.

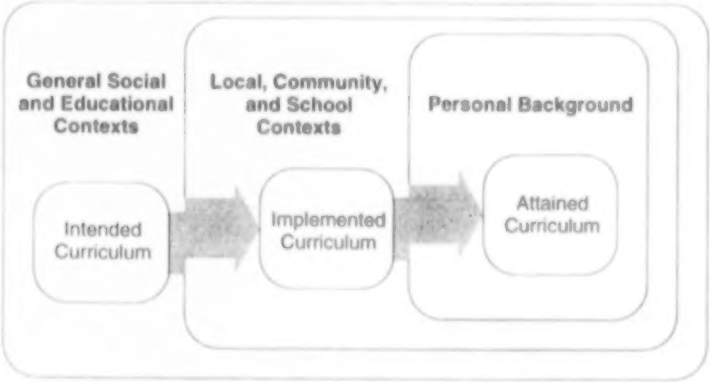

Figure 1. Conceptual framework for TIMSS

TIMSS was designed to address four fundamental research questions:

- How do countries vary in the intended learning goals for mathematics and science; and what characteristics of educational systems, schools, and students influence the development of these goals?
- What opportunities are provided for students to learn mathematics and science; how do instructional practices in mathematics and science vary among nations; and what factors influence those variations?
- What mathematics and science concepts, processes, and attitudes have students learned; and what factors are linked to students' opportunity to learn?
- How are the intended, the implemented, and the attained curricula related with respect to the contexts of education, the arrangements for teaching and learning, and the outcomes of the educational process?

The results released to date have focussed on the first three research questions, and a lot of the discussion about those results has concentrated on what are sometimes called the "horse race" aspects of such studies: Who won? The more important analyses—for example, analyses that will help us to make decisions about what combinations of educational inputs and processes are most consistently associated with high achievement levels—remain to be done, and funding to support those kinds of analyses is currently being sought.

DESIGN OF TIMSS

Three populations of students were identified for participation in TIMSS. Population 1 consisted of all students in the pair of adjacent grades that contained the majority of 9-year-olds: Grades 3 and 4 in most countries. Population 2 consisted of all students in the pair of adjacent grades that contained the majority of 13-year-olds: Grades 7 and 8 (*Secondaire I and II* in Québec) in most countries. Population

3 consisted of all students in the last year of secondary school, regardless of the type of program in which they were enrolled.

In almost all countries, children begin formal schooling at the age of six; and, at least in the industrialized countries, virtually the entire age cohort of children remains in school through age 16. Therefore, the national samples for Populations 1 and 2 should be comparable across countries. Comparability at the Population 3 level is more problematic because of significant differences among countries in the proportion of students remaining in school until the age of 18, in the structure and content of school programs at the senior secondary level, and in the extent to which students are encouraged to specialize in one or more subject areas.

Three distinct groups were sampled in Population 3. The mathematics and science literacy sample consisted of students who were in their last year of secondary school, regardless of the type of program they were following. The advanced mathematics and the physics samples included only students who were taking advanced courses—as defined by each country—in either or both of those disciplines.

Countries were encouraged to make their Population 3 definitions as inclusive as possible in order to maximize the comparability of the results. To help assess the degree to which the samples were comparable across countries a *TIMSS coverage index* was defined. The coverage index is an estimate of the percent of the school-leaving age cohort covered by a country's Population 3 sample. The denominator of this statistic is the number of students in the national school-leaving age cohort. The numerator is the number of students included in the national definition of Population 3. Three coverage indices were calculated: one for mathematics and science literacy, one for advanced mathematics, and one for physics.

In Canada, except for Québec and Ontario, Population 3 was defined to include all students in Grade 12, and the mathematics and physics specialist sub-populations were defined in terms of particular courses in each of the provinces. In Ontario, the Population 3 samples consisted of students completing OAC programs, some of whom were in Grade 12 but many of whom were in Grade 13. In Québec, the Population 3 samples were selected from students enrolled in second- or third-year CEGEP programs.

Nationally representative samples of Canadian schools and classrooms were selected by Statistics Canada, and cooperation from the schools was extremely high for Populations 1 and 2 and the two specialist sub-populations at Population 3. The cooperation rate for the mathematics and science literacy study fell slightly below the criterion level of 85 percent established for the study.

Data collection was carried out in the spring of 1995, and some 500,000 students, their teachers, and principals around the world participated. Students wrote achievement tests that had been developed specifically for use in the study. They included both multiple-choice and constructed-response items, the latter requiring hand scoring. Students also completed a questionnaire designed to solicit information about their backgrounds, opinions, and attitudes. A sub-sample of students in Grades 4 and 8 also participated in a survey of hands-on problem solving.

Principals completed a school questionnaire, and teachers provided information about their personal and professional backgrounds, teaching practices, and coverage of the curriculum. A detailed analysis of curricula in mathematics and science was also carried out. The relationship of each of these components to the conceptual framework of the study is summarized in Figure 2.

THE ACHIEVEMENT RESULTS

The first round of analysis of the TIMSS data has focussed on descriptive analyses of the achievement results and the data from the student, teacher, and school questionnaires. Four reports have been published to date: one for each population and one for the hands-on problem-solving component.

An overall view of the achievement of Canadian students is provided in Figure 3. The chart summarizes the achievement of students from the G–8 countries relative to the international mean for each population on the seven achievement tests that were administered.

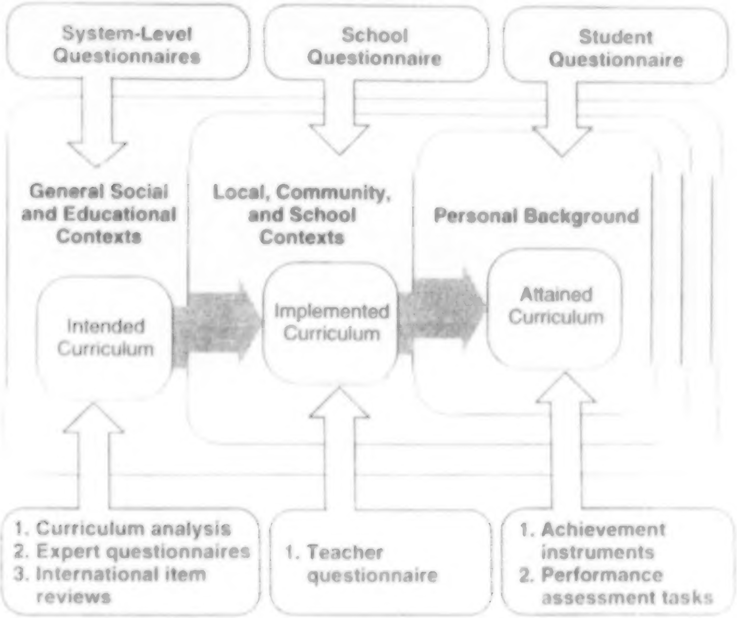
System-Level Questionnaires
School Questionnaire
Student Questionnaire
General Social and Educational Contexts
Local, Community, and School Contexts
Personal Background
Intended Curriculum
Implemented Curriculum
Attained Curriculum
1. Curriculum analysis
2. Expert questionnaires
3. International item reviews
1. Teacher questionnaire
1. Achievement instruments
2. Performance assessment tasks

Figure 2. Relationship of TIMSS data collection instruments to the conceptual framework

Many countries did not test as large a proportion of their Population 3 students as Canada did, especially in advanced mathematics and physics. Russia, for example, had coverage indices of only two percent for advanced mathematics and physics, while Canada's were 16 and 14 percent, respectively. Because of the wide variability in coverage indices among countries with respect to the Population 3 specialist sub-populations, Figure 3 compares the performances of the top five percent of students only. (The Population 3 circles in the chart for Russia are smaller to indicate their low coverage index.)

Canadian students scored significantly higher than the international mean on five of the seven tests: Grade 4 (Population 1) science, Grade 8 (Population 2) mathematics and science, as well as advanced mathematics and mathematics and science literacy for Population 3. None of the Canadian scores was significantly lower than the international mean. The overall impression conveyed by these results is that the achievement of Canadian students compares very favorably with that of students from the major industrialized countries. It is unfortunate that not all countries participated at each population level, and there is little doubt that Japanese students would have achieved at a level significantly higher than the international mean for Population 3 had they done so.

Figure 4 presents a corresponding summary of the within-Canada results for those provinces that selected large enough samples to produce stable estimates of performance at the provincial level. (The there is little doubt that Japanese students would have achieved at a level significantly higher than the international mean for Population 3 had they done so.

Figure 4 presents a corresponding summary of the within-Canada results for those provinces that selected large enough samples to produce stable estimates of performance at the provincial level. (The Québec samples were also large enough for this purpose; however, the agreement with the Ministry of Education in that province stipulated that Québec was to be included only in the national sample.) The chart shows considerable variation in performance among the provinces, with Alberta and British Columbia having the best performance overall.

	Gr 4 Math	Gr 4 Sci	Gr 8 Math	Gr 8 Sci	Pop 3 Math (top 5%)	Pop 3 Physics (top 5%)	Pop 3 Literacy
CANADA	○	◍	◍	◍	◍	○	◍
England	●	◍	○	◍			
France			◍	●	◍	●	○
Germany			○	◍	●	○	○
Italy					●		●
Japan	◍	◍	◍	◍			
Russia			◍	◍	◍	○	●
United States	◍	◍	●	◍	●	●	●

◍ Significantly higher than the international average

○ Essentially the same as the international average

● Significantly lower than the international average

Figure 3. Canadian performance at three population levels, relative to other countries

	Gr 4 Math	Gr 4 Sci	Gr 8 Math	Gr 8 Sci	Pop 3 Math (top 5%)	Pop 3 Physics (top 5%)	Pop 3 Literacy
CANADA	○	◍	◍	◍	◍	○	◍
British Columbia	○	◍	◍	◍	◍	○	◍
Alberta	◍	◍	◍	◍	◍	○	◍
Ontario	○	◍	○	○	◍	○	◍
New Brunswick	○	◍	○	○			○
Newfoundland	○	◍	○	○			

◍ Significantly higher than the international average

○ Essentially the same as the international average

● Significantly lower than the international average

Figure 4. Performance of Canadian provinces at three population levels

Canadian students' scores were in the top third of the range of scores on each test, and, in most cases, in the top quarter. Table 1 shows, for each test and each population, how many countries had scores that were either significantly higher or significantly lower than Canada's. Using this as a standard indicates that Canadian students had their best results in Grade 4 science and in Population 3 advanced mathematics. Their weakest performances were in Grade 4 mathematics and Population 3 physics.

Table 1. Summary of the relative performance of Canadian students in TIMSS

	No. of countries participating	No. significantly higher than Canada	No. significantly lower than Canada
Grade 4 Mathematics	26	11	10
Grade 4 Science	26	5	13
Grade 8 Mathematics	41	9	19
Grade 8 Science	41	12	15
Pop 3 M&S Literacy	21	3	10
Pop 3 Adv. Math. (5%)	16	3	6
Pop 3 Physics (5%)	16	6	4

It is not possible to be highly precise about these results. Many of the media reports published immediately after the results were announced gave the countries ranks on each test. This is inappropriate given the small differences in mean scores frequently found between countries with adjacent, or nearly adjacent ranks, and the sizes of the standard errors. On every test at least one quarter of the countries had scores that were not significantly different from Canada's, and ranking makes little sense in such circumstances.

QUALITY ASSURANCE MEASURES

Critics of international studies comparing students' achievement have criticized previous studies for a number of real or potential weaknesses in design, execution, and analysis. In the case of TIMSS, a great deal of time, energy, and resources were devoted to ensuring that the highest standards were maintained in every major aspect of the study (see Martin and Mullis, 1996).

A major effort was devoted to ensuring that the national samples selected in each country were representative samples of all students at those grade levels in that country. Strict standards were established and a team of sampling experts (including personnel from Statistics Canada) oversaw every stage of the sampling process. Countries whose samples were found wanting in some respect either had their results flagged in the international reports; or, in cases where the problem was deemed sufficiently serious, their results were not published.

Similarly high standards were established and followed in a number of other areas. The test items used to evaluate students' achievement were developed by experts in mathematics and science education from around the world. Several rounds of pilot testing of the items were carried out, and a complete field trial of all the instruments was done in every country. Strict standards for translation and translation verification were established, and an independent contractor verified a sample of translations from each country. Training sessions were held for national coordinators as well as for those responsible for coding students' responses to the constructed-response items. Independent referees were employed and trained to visit a sample of participating schools in each country to ensure that recommended procedures were being followed. Data cleaning, verification, and analysis were carried out in international centers with records of expertise in doing such work.

Some initial reaction within Canada to the results published so far has focussed on a concern that the tests may have been more appropriate to the curriculum in some provinces than in others. The only data available on the appropriateness of the tests to provincial or national curricula comes from a survey of a small number of people in each province and country. They were asked to examine each item and state whether or not, in their opinion, the item was appropriate for that grade level in their jurisdiction. The small number of respondents and their unknown degree of familiarity with the specifics of the curriculum demands for that grade make the reliability of that data questionable.

On the other hand, the results of a test-to-curriculum matching analysis show that, in all countries and for all three populations, the mean scores and the rank order of countries changed very little regardless of which subset of items were used. The analysis was based on that subset of the item pool which a given country had deemed appropriate to its curriculum. There was a *Japanese test* consisting of the items deemed appropriate by Japanese educators, a *Canadian test* consisting of the subset of items deemed appropriate by Canadian experts, and so on. National means were calculated for each of these "tests," and the overall finding was that there were few changes either in the mean score on these national tests compared to the scores on the international test or in the rank order of the countries on those tests.

CONCLUSION

TIMSS is the first international, comparative study of educational achievement in which representative samples of Canadian students from public and non-public schools, as well as both English-speaking and French-speaking schools have participated. Five provinces—British Columbia, Alberta, Ontario, New Brunswick, and Newfoundland—selected samples that were large enough to enable them to make comparisons with other provinces and countries. The achievement comparisons were based on tests covering a broad range of topics in mathematics and science curricula.

The results of the achievement comparisons show that Canadian students are learning a lot of mathematics and science in school, and they are able to apply those concepts and skills in solving problems. Students from some countries—especially Asian countries—consistently outperform students from other countries including Canada. There is considerable variability in the results at the provincial level, with Alberta and British Columbia getting higher scores fairly consistently. That level of performance shows that Canadian students are capable of obtaining scores as high as those of students anywhere.

There is a lot of good news in these results, and parents and taxpayers should take some comfort from them. If we were giving grades to the TIMSS countries for their performance, Canada would certainly deserve a strong "B;" and Alberta, an "A-."

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Topic Session 3

THE COMMONSENSE OF TEACHING

David Wheeler
Concordia University, Montréal (ret'd.)

This talk was in Cassandra mode. You know the sort of thing: society is crumbling, war and pestilence are around the corner, the situation is hopeless, and similar animadversions. Those of a nervous disposition would be well advised to stop reading now and take a calming stroll.

When I first thought about giving this talk, I intended to give most of the time to technical aspects of teaching, especially the teaching of mathematics, a topic which doesn't seem to me to get the detailed attention and study it deserves. In starting to work on the talk, however, I found a lot of more general issues about teaching came into my mind that couldn't be pushed aside. What follows is mostly this other "general stuff", so the paper should now be read as a prelude to the one I had originally hoped to present.

A second "caution to the reader" may be in order. I open with some "negative thoughts" that appear to be dismissive of a great deal of very dedicated work by many teachers, teacher trainers, innovators, and researchers. I start with these because they indicate the position of the "frame" through which I am looking at questions in the field. The frame is deliberately placed to emphasize how much further remains to be travelled than the distance we have come. It insists on reminding us that we have hardly begun to articulate and communicate the skills that underlie good teaching. This viewpoint was important for my purpose in giving the talk, which was to provoke my listeners into thinking as much about teaching as they do about learning. At other times I have other purposes, or I am talking to other audiences, and then I adjust the frame accordingly.

NEGATIVE THOUGHTS

I took up my first teaching appointment, in a high school, in 1947 and last year I gave up the editorship of *For the Learning of Mathematics*, so I've been involved in one way or another with the teaching of mathematics for 50 years. What do I see now when I look at teaching from the perspective of this long haul?

- Almost all teaching is amateurish.

Amateurs may love their work, as the etymology of the word suggests, but society expects professional expertise from its teachers. I see some very effective teachers, but I also see many who don't seem to have the resources of skill and know-how needed to teach effectively in the difficult circumstances that many schools present. I have little sense that there is agreement among teacher trainers about the technical and/or professional equipment a teacher needs, and some of the people training others to teach seem to doubt whether teaching involves the application of any techniques at all.

- Almost all that is said about teaching is banal.

This thought reinforces the first. The banality seems to arise from uncertainty about the basic requirements for effective teaching. Because we haven't resolved the matter of what comprises the basic equipment, conversation about teaching never goes much beyond discussing "starting points" and we hardly ever get to work on the more searching and sophisticated questions that classroom practice throws up. The claim that "teaching is an art" can too easily become an evasion of responsibility.

- Though teachers review and reflect on their actions, they almost never reflect on their beliefs.

This is a tricky point. Western societies, in general, permit people to believe what they want—it's one of the pillars of a free society. The downside to this freedom is that people begin to think that beliefs don't have to be checked out, that evidence for or against their validity doesn't have to be considered, that the only authority a belief requires is that enough people hold it. Teachers, whose beliefs affect what they do, and whose beliefs may not be entirely compatible with actions the educational system tells them to take, need to be particularly alert to both the overt and covert effects of the beliefs that touch on their work in their classrooms.

- Almost all educational trends are essentially concerned with reinventing the wheel.

Re-inventing the wheel, in spite of the old jibe, isn't altogether a bad thing to do. Each generation or two of teachers meets fresh educational and social problems, or old problems with a new twist, and a re-inspection of the ways they are being handled can prove useful. But of course this point highlights the lack of a well-founded *tradition* of good teaching practices and of a suitable machinery for inducting new teachers into it. (Our wheels remain square, one could say, and we continue to find them unsatisfactory, rolling them first this way, then that, to no appreciable advantage.)

N.B. The above statements are not, of course, for general circulation! Should any parent or politician accuse me of uttering them I shall immediately deny that I made any such observations.

The following four sections offer my choice of themes connected to teaching that I put forward as worth thinking about.

PARADIGMS

Here I insert a point about the irreducible components of a model of teaching, and to suggest other paradigms of asymmetic social interaction that can be usefully compared with and differentiated from teaching. (In all that follows I am chiefly thinking of the teaching that takes place in institutional settings—"classroom teaching.")

French didacticians have helpfully focused attention on the centrality of the triad: *teacher/student-/subject matter* and have framed many of their empirical studies to clarify the interactions among its components. In this form, however, the triad makes no explicit acknowledgment of the culture (in all the large and the small senses of that word) within which the triad is situated. The effect of this culture (or of these cultures, because the classroom is the meeting ground of a number of independent and sometimes incompatible cultures) is easily overlooked, yet it seems plausible to me that in institutional teaching the cultural factors embedded in the teaching environment—the customs, values, expectations, etc., particular to the systems involved—have effects that always influence, and sometimes dominate, the interactions among the elements of the triad. Taking account explicitly of this "fourth party" can safeguard us from painting an unrealistic picture of practical pedagogical possibilities.

The irreducible components of a model of teaching, I suggest, are: *teacher/student/topic/context*, where "context" covers all of the physical, linguistic, social, and cultural attributes of the place where the teaching happens. The French model assumes, metaphorically speaking, two actors with a text, and I am suggesting it is important to integrate these into the "setting"—the theatre itself, the type of stage, the scenery, the audience's expectations.

Are there insights to be obtained by comparing the "teaching paradigm" to others? Here are a few pairings to consider.

- Teacher / student
- Craftsman / apprentice
- Mother / infant
- Guru / disciple
- Coach / ball player
- Counsellor / client
- Abuser / victim

In many ways the last six can be regarded as variants of the teaching paradigm. I include the very last one as a hint to you to entertain the idea that teaching doesn't always have positive and liberating effects. Other pairings will probably occur to you.

Having noted some similarities between the paradigms, we can then try to identify what, if anything, is special to the first. This may help us become clearer about what actions properly belong to teaching and what prevents it from slipping or sliding into one of the other related but different activities.

Perhaps, too, it is worth considering the ways people learn how to play their parts in these various activities.

THEORIES AND SUCH

The following is the introduction to a paper in the Fall 1995 issue of the journal *Daedalus*.

> "Two challenges face American education today: 1) raising overall achievement levels and 2) making opportunities for achievement more equitable. The importance of both derives from the same basic condition—our changing economy. Never before has the pool of developed skill and capability mattered more in our prospects for general economic health. And never before have skill and knowledge mattered as much in the economic prospects for individuals. There is no longer a welcoming place in low-skill, high-wage jobs for individuals who have not cultivated talents appropriate to an information economy. The country, indeed each state and region, must press for an overall higher level of such cultivated talents. Otherwise, we can expect a continuation of the pattern of falling personal incomes and declining public services that has characterized the past twenty years.
>
> The only way to achieve this higher level of skill and ability in the population at large is to make sure that all students. not just a privileged and select few, learn the high-level, embedded, symbolic thinking skills that our society requires. Equity and excellence, classically viewed as competing goals, must now be treated as a single aspiration." (Resnick 1995)

The author's main argument in the paper is that the belief that *aptitude* is the chief determinant of educational success has unfortunate consequences; in particular, it discourages students from attempting "to break through the barrier of low expectations." She recommends a shift to an emphasis on *effort*, based on the assumption that "effort generates aptitude."

In this paper Professor Resnick makes the telling point that many of the practices in American schools enshrine the claim that aptitude is the most important factor in educational success, and that the effects of these practices covertly reinforce the belief even after public or professional opinion has (overtly) moved away from it. Probably the most striking example of such a practice is the use of SAT scores, which aim to be "knowledge-free," as important indicators for college admission. Could anything be more absurd than failing to consider the knowledge that 18 year old students have already acquired when deciding whether to accept them for further education?

The status of the proposition "aptitude determines success" appears to be that of a *theory* in the field of education, but we can remark that the concepts it deals with are not entirely clear, and the assertion doesn't seem to have clearly articulated connections to other theoretical statements. Does the proposition have empirical support? Can it defeat arguments attempting to disprove it? Perhaps it's an item of *folklore*, not of theory. Either way, Resnick reminds us, too many institutionalised practices which may at some time in the past have been derived from the "theory" now serve as at least a partial substitute for it, extending its life invisibly.

We need much more than this single instance to establish a significant generalisation, but it nevertheless triggers in me two small "lemmas":

- Educational "theory" doesn't govern educational "practice" in a straightforward way.
- What educators say they believe about teaching doesn't necessarily match the beliefs embedded in their practices.

Before quitting this example, I must express my extreme disquiet with Resnick's strategy. She expresses the basic options in simplistic either/or terms—"aptitude or effort," "equity or excellence"—as if her readers would be unable to appreciate a more nuanced account of the complexities she is dealing with, and her introduction is as shocking in its use of crude generalisations based on unexamined assumptions as anything in the beliefs and practices she criticises.

For my second example I go back over two hundred years to Britain. In 1749 David Hartley presented a systematic formulation of his psycho-philosophical theory of *associationism*. The philosophies of Locke and Hume were among its influences, J. S. Mill and Spencer among its later adherents.[1] The theory (which I simplify and abbreviate considerably) held that:

- Ideas and sensations are reflections of external objects.
- Particles transmit vibrations from the objects through the senses to the nerves and the brain. (This description of the material connection between object and sensation was later modified and eventually abandoned.)

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[1] I am indebted to Brian Simon's paper (1985), *Samuel Taylor Coleridge: The education of the intellect*, for triggering my thoughts about this example and for some of the detail in my account of it.

- "Complex" ideas are associations of simpler ones, which connect according to laws of simultaneity, contrast, contiguity, etc.

The first thing this theory does is bring the study of thought, reason, and other mental processes, into line with the cosmological and scientific beliefs current in its time: that (physical) phenomena are governed by fixed (mechanical) principles whose laws can be discovered through a combination of observation and insight. In the second place, it holds out the possibility that particular ideas can be generated in the mind by an appropriate manipulation of those immediate objects belonging to the material world. This latter implication was seized with enthusiasm, suited as it was to an intellectual climate much concerned with exploring the possibilities of secularising and democratising society. Among those influenced by the theory were the early utopian socialists, exemplified by Robert Owen and his slogan, "Circumstances make man!"

Associationism is essentially a theory of learning and when interpreted within a determinist frame of reference it proposes the possibility of making education truly scientific, producing guaranteed effects in learners by controlling in detail their educational environment. The theory generated a quite extraordinary mood of pedagogical optimism.

Coleridge and Marx, both originally highly sympathetic to the theory, came to acknowledge its drawbacks. Coleridge allows that associationism explains very clearly why certain items of knowledge are easy for us to retain and recall, but beyond that it fails by treating the learner as properly passive, "a lazy looker-on." He points out that by an act of will anyone can arbitrarily give distinction to any item of knowledge whatsoever. Marx asks in what way the theory can explain how the *educator* comes to know what learning is desirable and hence what sort of learning environments to construct, and adds that, rather than being the creature of circumstances, man is someone who changes circumstances and in the course of that action changes himself.

This example draws our attention to the fact that an educational theory doesn't have to have a proven track record to be adopted with enthusiasm. Together with the previous example it should encourage us to be sceptical about the development of sound educational theories, and especially about the too hasty attempt to deduce practical consequences from them.

One is very tempted to say, "a plague on all your theories."

NEW FACTORS

I consider myself very lucky to have begun teaching in the immediate post-World War 2 years. I was energetic and enthusiastic, and so was the general mood in society at the time. Teaching looked a worthwhile thing to be doing, its importance wasn't questioned, and everyone was optimistic about what institutional education would eventually bring to all students in society, not just those who, because of their social class, or because they were deemed unusually "bright", had already been led to expect it. How different the mood seems now, fifty years later! Yet, I don't side with those who say there has been a substantial falling-off in what teachers and students achieve: the hard evidence is scanty and somewhat ambiguous. But I can't deny that the optimism of society has been replaced by pessimism and that the schoolteacher's job at the end of the millennium appears to be immeasurably more difficult than it was 50 years ago.

Without claiming to be able to give all the reasons for this shift, I draw attention to three major factors affecting the environment within which teachers now have to work.

A. Education started out as the education of an élite. This is still the only education that is done well. In élite education the teachers and students share a common culture—a culture with a common language, common values, common expectations. Teachers confronted with the demands of "education for all" are faced with the genuine difficulty of working effectively in and around distinct and often incompatible subcultures.

B. Teaching in classrooms used to be teaching "behind walls," physically and metaphorically. Classrooms have traditionally been spaces where teachers practised their instructional expertise unquestioned by their peers and unobserved by all except their students. To an extent the walls have now been toppled. Teachers today are evaluated by their students, held accountable by their employers, and generally subjected to intense public scrutiny and pressure.

C. Computers and the Internet pose threats, real and imagined, to traditional teaching methods and to the traditional instructional vehicle, viz., books.

No wonder classroom teaching now seems so much more difficult and so exhausting!

It's highly unlikely the challenges now facing teachers can be met by trying to go back, to reassert the methods and values of the recent past. But the past is enormously powerful. Large educational systems have an intrinsic inertia now reinforced by the new openness of the system to outside criticism and influence. Parents and politicians even more than teachers may seek security in the familiar and shrink from radical change. How else can one account for the widespread popular rejection of the electronic calculator as a tool for teaching arithmetic in elementary schools? The rejection seems the expression of a fervent wish that the calculator didn't exist, had never been invented, so everyone could go on teaching as they did before it arrived. But the calculator *does* exist, the calculator *has* been invented, and there is no way the teaching of elementary number operations can or should go on as before.

THE PRACTICE OF TEACHING

Maybe the need now is for more creative and radical kinds of pedagogy. First, though, in this new "open" climate, we have to try to establish that pedagogy is *important*, that it's not just an academic word for something trivial, like knowing something and then telling or showing it to someone else. Teaching mathematics to all students is difficult, not because mathematics is particularly difficult, nor because students are, but because we now have to be able to teach it in such a way that *anyone* can access it if they need it and if they want to, and this is a requirement that can't possibly be met without the aid of a skilful and quite sophisticated pedagogy. We need to stop talking as if *teaching is an art*, which is only a sly way of saying "a few can teach, most can't," and as if *teaching is a science*, which would require a consensus about basic theories which won't be achieved for a long while, if ever, and settle for teaching as essentially a technical matter—not in the sense of a full-fledged technology but as a set of know-hows, a sort of kitbag for dealing with the practical demands of the classroom, a kind of *bricolage*.

Because I have spent so much time on this occasion talking about other issues, I can't go on here and now to give genuine substance to the previous sentence, though that is what I hope to begin to do another time. It's also my hope that some among my listeners and readers will also want to work in detail on this question of the technical nature of teaching.

POSTSCRIPT

In spite of the low esteem it suffers, and the institutional restrictions that confine it, teaching remains a wonderfully worthwhile activity. Just occasionally it yields the reward that is above all other—the awareness that for a particular student *intelligence has been revealed to itself.*[2] To make "revealing intelligence to itself" an explicit target may be unwise since we can so rarely be sure whether, how, or when the target has been attained, but as a general orientation, a vision we store in the back of our mind, it may ready us to seize any opportunity the classroom does offer to bring this gift to our students.

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[2] I have lifted the phrase from the sentence, "The problem is to reveal an intelligence to itself," on page 28 of Jacques Rancières (1991) *The Ignorant Schoolmaster: Five Lessons in Intellectual Emancipation.*

AD HOC SESSIONS

Adhoc Session 1

A LOOK AT THE CONFERENCE LOGO

Klaus Hoechsmann, University of British Columbia, Vancouver

On May 29, as I rode on the bus with my new CMESG/GCEDM Conference bag, a woman across the aisle from me was staring at the logo as if trying to decipher it. Feeling uneasy about this undeserved attention—I could not have told her anything intelligible—I decided to make amends and prepare an *ad hoc* presentation for the next day: the usual story about Fibonacci and the Golden Section, pentagons and spirals, but—in honor of the unknown woman—using neither *algebra* nor *ratios.*

Perhaps she was an engineer or an actuary, but if her mathematical career was closer to average, she had probably been left befuddled by fractions, percentages, rates, ratios—especially irrational ones—and if she had cleared that daunting set of hurdles, chances are that she finally gave up when she ran into algebra. So let us shift to a perspective she might share.

Given a rectangle R, let us make a bigger one R'—and refer to it as the *augmentation* of R—by attaching a square to the longer side of R. Further, let us say that R is *golden* if R and R' are *similar* (i.e., can be lined up with sides and diagonals pairwise parallel). Easy exercises: golden rectangles are readily constructed by just compass and straight-edge; if R is golden, so is R'; and a golden rectangle cannot be tiled by squares—which means, its sides are *incommensurable.*

The deviation from goldenness of an arbitrary rectangle R can be measured by a piece of area we might call the "defect" of R. Then Fibonacci's rabbits are outflanked by the following theorem: *any rectangle R and its augmentation R' have the same defect.* Since repeated augmentation creates ever larger areas, this defect grows ever more tiny by comparison—making successive rectangles become more and more golden. And if you start with a unit square, their sides will measure 1, 2, 3, 5, 8, 13, . . .

Our small *ad hoc* group dwelt so long on these details that little was said about pentagons and spirals—except that the former are made from golden triangles (rectangles with one side collapsed), and the latter can never be composed of circular arcs (appearance notwithstanding) if they are to emulate nautilus shells.

After the seminar, I finally took a good look at my bag and made an embarassing discovery: the diagram on that logo was not golden at all—but "bronze" at best—with each "augmentation" adding only one half of a full square, hence Fibonacci's sequence replaced by 1, 3/2, 7/4, 19/8, 47/16, 123/32, . . . with quotients converging to 1/4 plus root 17/16. No wonder the lady was intrigued!

Adhoc Session 2

PRESERVICE PRIMARY TEACHERS' BELIEFS ABOUT MATHEMATICS AND LEVELS OF COGNITIVE FUNCTIONING

Sandra Frid
University of New England, Australia

This study was concerned with examining concurrently affective and cognitive aspects of pre-service primary teachers' mathematics learning. Its roots are in three research sources: constructivism, beliefs about mathematics and mathematics teaching and learning, and the SOLO model of cognitive growth (Biggs, and Collis, 1991). Specifically, the study aimed to: (i) examine students' beliefs about mathematics, mathematics learning and mathematics teaching, and (ii) determine students' cognitive levels according to the SOLO taxonomy.

A sample of 74 out of 140 first year students in a 3-year Bachelor of Teaching program completed a beliefs questionnaire and the Collis-Romberg Problem Solving Profiles (1992) at the beginning and end of semester. During this semester the students were enrolled in a first course in mathematics education. In classes of 25-30, they were provided with opportunities to work in collaborative groups, to explore different problem-solving approaches and to focus upon conceptual understanding. The mathematical explorations were aimed at enhancing their knowledge of mathematics and the teaching of mathematics, but were also intended to challenge their conceptions of mathematics and the nature of mathematics learning. Small and whole class discussions were used to focus on a teacher's role within these learning activities. Thus, the teaching approach was compatible with constructivism. Near the end of semester, 15 students participated in semi-structured interviews designed to broaden and elaborate the questionnaire findings.

In general, it was found that students did not hold the stereotyped views of mathematics that the research literature implies to be the case (e.g., rules and procedures, right or wrong, and only one way to do things). Students' beliefs at the end of semester shifted towards being even less stereotyped. For their beliefs on mathematics teaching, students generally simultaneously held 'constructivist' as well as more 'traditional' views, and these shifted slightly towards constructivism at the end of semester. Similar findings emerged for studentsÕ beliefs about mathematics learning.

On the problem solving profiles, students were very 'scattered' in that it was often difficult to assign an overall specific level for a student. Further, there was a slight drop in level at the end of the semester. These results were surprising and generate numerous questions, including: How valid are the profiles for this type of student (i.e., non-school students)? How 'seriously' do students take a problem solving test that does not 'count' towards grades? Is there any relationship between students' levels of cognitive functioning in mathematics and their beliefs about mathematics?

What the findings of the study imply that is of relevance to future research and teaching with mathematics teachers is that further consideration needs to be given to the notion of a continuum, rather than a duality, of constructivist versus more traditional views of mathematics teaching and learning, and how such a continuum can be informative to curriculum development in mathematics education.

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Ad Hoc Report 3

WHERE DO I WANT STUDENTS' ATTENTION? AND WHAT CAN I DO TO AFFECT THEIR ATTENTION?

Dave Hewitt
University of Birmingham, UK

Following John Mason's lecture, we explored further the issue of where attention is placed and reasons why a teacher might want to affect where a student places his/her attention. Furthermore, we considered articulating techniques which might be employed to help affect students' attention.

We began with the following written on the board: $5 + 5 =$

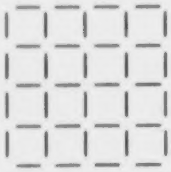
We discussed different consequences of what is stressed and what is ignored. For example (underlining indicates stressing):

$5 + 5 = \underline{5} + 4 + 1 = \underline{5} + 3 + 2 =$

$5 \underline{+} 5 = 6 \underline{+} 4 = 7 \underline{+} 3 =$

$\underline{5} + \underline{5} = 2 \times 5$ (2 lots of 5). A shift of attention can then result in $2 \times 5 = 2 + 2 + 2 + 2 + 2$

We looked at the different ways of stressing parts of the following drawing when given the task of counting the numbers of 'matches' involved:



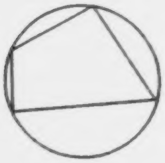
This resulted in statements such as the following:

$2n(n + 1) =$ $n(2n + 1) + n$ $2n^2 + 2n$ $4n + 2n(n - 1)$

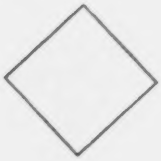
These resulted from stressing what calculations each person was carrying out to find the number of 'matches', rather than actually carrying out any calculations. The stressing of the process rather than the answer shifts attention from arithmetic to an algebraic structure, which can then be expressed as an algebraic statement.

The following was then put on the board and people were asked to consider, if they were teaching something using these images/symbols, where they would wish students' attention to be:

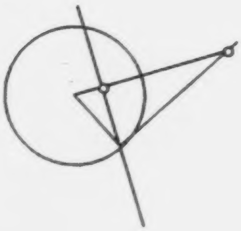
A



B



C



D 200 + 300 E x

This was followed by considering what might be done to try to focus a student's attention onto the aspect chosen in each case. I will offer two examples of what was offered:

For B: rotating the figure so that one of the sides became horizontal, and then rotating it back to its original position (attempt to shift attention onto the property of squareness when such a figure is often not considered to be square by students).

For D: stressing aspects of the number names: two *hundred* plus three *hundred* (attempt to place attention on the value aspect of the number names in an attempt to help students use an existing number 'fact', 2 + 3 + = 5, in this new situation).

I leave the reader to consider how they would approach other images.

Ad Hoc Session 4

IMPACT MATH: A MATHEMATICS REFORM PROJECT FOR ONTARIO GRADE 7 AND 8 TEACHERS

Doug McDougall
University of Toronto

This presentation highlighted the "train the trainer" model used in the implementation of the new Ontario Grades 7 and 8 Mathematics curriculum and the structure used to support teachers as they implement this new curriculum.

In June 1997, the Ontario Ministry of Education and Training (MET) released the new Ontario Curriculum, Grades 1-8: Mathematics, 1997. This document was developed to "provide a rigorous and challenging curriculum for students in Grade 1 to Grade 8 (MET, 1997, p. 3)." The curriculum expectations for this new curriculum document is much more rigorous and demanding than previous curricula. This document is in much more detail than its predecessor, eliminating the need for individual school districts to write locally defined expectations for students at every grade level.

A "train the trainer" model was used to introduce the new material to teachers and build capacity in schools and school districts to use the new curriculum. The cadre of ten trainers each conducted two 2-day workshops at sites throughout Ontario. The over 400 participants and the cadre of trainers will serve as a base of expertise, dispersed throughout the province to provide resources and support in subsequent implementation efforts locally.

The Impact Math implementation plan encourages participants to explore mathematics. It assumes that teachers will gain a better understanding of their own and children's understanding of mathematical structures and relationships through this exploration (McDougall, 1997). This approach has been encouraged by numerous researchers (Ball, 1997; Remillard and Geist, 1998). The Impact Math Project also provides teachers with sample student activities, including student work using a four level rubric for scoring. These activities were field-tested and samples of student works were selected to illustrate the rubric defined by the Ministry of Education and training for mathematics classroom assessment.

Curriculum reform in mathematics can be used as an opportunity to reform mathematics teaching practice. In Ontario, the new Grade 1 to 8 mathematics curriculum was introduced without an implementation plan by the government. The "train the trainer" model is being used to implement the curriculum. The project will be evaluated to determine if this method can lead to change in large scale implementation programs.

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Ad Hoc Session 5

A MAN LEFT ALBUQUERQUE HEADING EAST WORD PROBLEMS AS FOUNDATIONAL NARRATIVE IN MATHEMATICS EDUCATION

Susan Gerofsky
Simon Fraser University, Vancouver, BC

> A rock dropped from the top of the Leaning Tower of Pisa falls 6 m from the base of the tower. If the height of the tower is 59 m, at what angle does it lean from the vertical?
> - Ebos et al. (1990). Math matters: Book 3. Scarborough: Nelson Canada. p. 354.

Word problems establish a foundational narrative in our mathematical education from the earliest years of primary schooling. Obliquely, through a seemingly inexhaustible repertoire of odd, inconclusive stories (like the one cited above), word problems let us know what mathematics is allowed to be about, how we are to go about doing mathematics, and particularly, what is the uneasy relationship between the world of mathematics and our lived lives.

Word problems are stories, but very thin stories—poor quality fiction at best, with no plot or character development, minimal setting and action, and little insight into important issues of what it means to be human. They appear to be about real people, places and things ("the Leaning Tower of Pisa", "Albuquerque") but they are not really about those things. Rather, they are ambiguous, referring in a coded way to the world of mathematical objects and processes through apparent references to the world of human experience, although a few points of contact remain with things as we know them in everyday life. The contingencies of lived experience do not apply here; the student who knows too much about the purported topic of a word problem would do well to leave that knowledge behind when entering the mathematics classroom.

Word problems are mathematics "dressed up" as stories—or are they stories that, when heaped up by the score, embody mathematics?

This presentation consisted of a reading in the form.of a dialogue about mathematical word problems as story, as language, as pedagogy. It is a play on the words and forms of this 4000-year-old educational genre. A man left Albuquerque heading east gives voice to possible meanings of word problems as riddles, parables, cartoons, the folktales of a shared mathematical culture. The audience was invited to hold onto the ambiguity of this curious and long-lived pedagogical narrative.

NEW PhDs: RESEARCH REPORTS

PhD Report 1

LEARNING ALGEBRA PERSONALLY

Ralph Mason
University of Manitoba

This presentation was a jazz composition, a story-telling as a weaving of three sets of voices with instrumental background. Foremost was the narrator's voice, re-telling a sequence of events from a strand of research in a grade nine algebra class. In the role of the chorus were the voices of critical theorists (see Appendix 1) and educational theorists (see Appendix 2). Quotations of their writing were dispersed among the presentation's audience, for interspersing when any audience member felt was appropriate. Background instrumentation included the use of algebra tiles and a series of worksheets for introducing polynomial arithmetic; a critical application of constructivist theory to guide small-group mathematical inquiry; and the use of journals, transcriptions, and retrospective conversations for viewing and re-presenting the actions and interactions. However, the voice that carried the melody was that of Benazhir, a student.

The research began as the design and implementation of a curriculum for the pre-algebra, functions and relations, and algebra of grade nine mathematics, to take place in two of every five mathematics periods per week throughout the year. The design depended on certain premises:

1. of algebra as more than any singular phenomenon (Kieran and Chalouh, 1993; Usiskin, 1988; Wagner and Parker, 1993). Algebra was considered a language for the expression of patterns within functions (Borasi, 1992; Goldberg and Wagreich, 1990; van Dormolon, 1989), a symbol system with its own syntax (J. Mason, 1989; Siegel, 1995; Smith, 1995), and a forum for generalizing and abstracting the structures of arithmetic (Lietzel, 1989; Sfard, 1994).

2. of constructivism as a descriptive theory of the cognition of learning, and of teaching as the causing of learning through activities and discourse with different forms, content, and roles (Adams and Hamm, 1994; Borasi, 1992; Romagnano, 1994; Smith, 1996; Wells, 1995). Perceiving our pedagogic goals to be the search for the one concrete representation or activity that captures the essence of a concept and portrays it for the students is simply a transference of our former beliefs in clarity and precision within the delivery metaphor, where just the right explanation at the right time delivered in the right way was believed to convey the essence of an idea to a learner. Instead, constructivism is better grounded in an epistemology where a person's understanding of any content is based on complex connectedness among elements and to elements of other content, with both the elements and the connectedness among them emerging through multiple and varied experiences of the learner. Our pedagogy must match not only the psychology which constructivism suggests but also its epistemology: if understanding is the meaningful interpretation of personal experience, then generalized understandings will depend on the meaningful interpretation of multiple experiences (Driver and Scott, 1995; Noddings, 1994; Steffe and Kieren, 1994; Thompson, 1988).

3. of teaching as the encouraging of students within pedagogic relationships (Jervis and McDonald, 1996; van Manen, 1994), building on their sense of their roles as learners and as students (Wilson, Peterson, Ball, and Cohen, 1996).

4. finally, of learning as an individual cognitive process and outcome of fundamentally contextualized and interpersonal processes (Cobb, 1995; Schifter, 1996; Steffe and Kieren, 1994; von Glasersfeld, 1991).

Fundamental to the teaching design was the expectation that any teaching procedure that anticipates changes in student roles and learner roles must incorporate the teaching (negotiation and development) of the new roles. It was in part this challenge that made necessary the multiple interaction formats among students and between students and the teacher-researcher. This included audio-taped conversations during and after classes, and a response-writing procedure that involved the teacher-researcher responding to each exit slip written by each student. All of this provided a significant flow of data, along with the mathematical products that the students generated and the exit conversations with each group of learners at the end of the year's intervention.

But what of Benazhir? Hers was a quiet voice, but not a silent one (Belenky et al., ,1986). By everyone's accounts, Benazhir was an excellent student, well-behaved, industrious, and reliable. This was especially apparent whenever Benazhir made up for her relatively frequent (two or three days out of every ten) absences: she always had a note from her mother, and always had her missed work caught up, and never used her absences as an excuse or as a reason for any special request. Benazhir was not a superstar, but she maintained honors achievement.

Benazhir enjoyed the small-group aspect of the new mathematics approach. It gave her a chance to interact quietly with her group-mates, two other reserved and dedicated students. Benazhir also found it useful to be able to ask her colleagues what precisely she was to do in each circumstance of an activity or an inquiry. Like Benazhir, her colleagues believed initially that being good at math was a matter of determining exactly what was expected, doing it precisely, and remembering to do it the next time those instructions were encountered. For instance, when learning to deal with algebra tiles, Benazhir paid close attention to the wording of instructions such as "Write in factored form" and "Factor completely" to decide if she should sketch algebra tiles or show her thinking.

During the course of the year's instruction, Benazhir's two colleagues changed their orientation to such instructions. They learned to make up their own minds regarding what would generate rewarding understandings of the subject of inquiry. They learned to value showing their thinking in multiple forms, and sharing their thinking with others. However, Benazhir did not. For instance, she did not find valuable any opportunities to discuss mathematical processes. Once, when a letter which was sent to parents suggesting that they could ask their son or daughter about the themes about learning math that had emerged in class, Benazhir wanted the suggestion removed from her letter. "We *don't* discuss math at home." When probed, she revealed, "My father doesn't talk to his daughters about *anything*. And my mom can't encourage me at all, except in cooking and looking after my sister and stuff, or my dad will get mad."

It is much more pleasant to document the changes and growth of Benazhir's classmates. As the students grew in their autonomy and voice, they also grew in their willingness and ability to engage in rich learning processes in mathematics. Yet Benazhir continued to do what she was told, and do it well. Yet, when asked whether reports to parents should mention the now-voluntary written reflections which her classmates were finding rewarding, she replied, "Don't put a lot of emphasis on it, because I found it quite annoying at times, when you asked insignificant questions or things that didn't relate to the topics or subjects of our classes." Even this bluntness was rare, and could be considered to be a result of the fact that reports to parents were an issue that Benazhir wanted to monitor closely. Somehow, it was as if

Benazhir did not want to develop autonomy or voice in terms of her learning at school. Did she want math to be about matching the correct rote procedure to the teacher's given instruction?

When her mathematics teacher and I were discussing the matter one day, it came to us what was happening. Benazhir was known to come from a devout Muslim family, although no one gave it much thought. Yet here was an answer to her enigmatic behavior: Benazhir was being groomed for a traditional role as a devout wife. As such, neither mathematics specifically nor extended success at school fit into her parents' plans for her. This made sense of both her frequent absences and her concern about any letters or marks sent home.

However, it also made sense of Benazhir's reluctance to adopt a richer approach to learning. If being successful meant engaging in rich activities and the ongoing and ensuing discourse about those experiences, then Benazhir would not be able to continue to succeed. Given her required absences, Benazhir could only continue to excel if excelling was a matter of doing a prescribed set of work precisely and diligently. The richer pedagogy threatened to leave her behind.

An explanation is not a solution, however. This sudden appearance of ethnicity and gender was a shock to me, and it offered no apparent avenue of involvement with Benazhir's dilemma. In fact, it was quite apparent that Benazhir carefully avoided any possibility of such involvement by any of her teachers, and any intrusion by me would be uninvited and unwanted. This was a case of honoring Benazhir's right to privacy: despite the impact of her family life on her learning of mathematics, her family life was not available to me either for further inquiry or for intervention. As we recognize the potential of noticing and relating to the whole persons whom we teach, and as we recognize the potential of guiding those persons to learn in richer and more personal and interpersonal ways, we will have to accept that this range of new possibilities also has limitations which we do not yet know how to cross.

One final voice emerged in this presentation. One audience member waited until a private moment, and asked why I had told such a sad story. "We need our courage, and I need stories of our successes as mathematics teachers for the courage to continue to stretch in my pursuits," I heard her say. I do not know what I said to her. I was distressed to have disturbed her. Yet somehow, I had known that telling Benazhir's story (and thus telling a story from the edges of my own significant success within this research) would be at least a sobering if not a disturbing thing. Can I answer her question now?

I will try. My telling of Benazhir's story is a telling about limits, not of tragedy. Benazhir's story brings to light the complexities of the challenges we face, and it brings into clear focus the importance of what we study and pursue. The reform and improvement of the teaching and learning of mathematics is significant, precisely because it is embedded in complex personal matters such as in Benazhir's life. In that way, I hope that readers of this version of the story can take heart in their endeavors as researchers and teachers and learners.

Just as Benazhir could not tell this story, I must. It is a story worth hearing and worth feeling, but first it is a story worth telling. For me it is in part a story of research ethics, of respecting limits to one's efficacy and desire to understand when intrusion would be wrong. It is a story of my learning about my own ethnicity and gender and its meanings to my research and teaching. That I did not presume to judge her chosen pathway nor to alter it is no dishonor. I cared for Benazhir, and cared about Benazhir, and I taught her well and ethically (Noddings, 1995). Benazhir honored me by sharing part of her life and part of her story and letting me share that part with others It is a story of how coming to care and coming to know our students co-emerge within our teaching and our research, bringing to view more than our interactions can control. I hope the story of Benazhir can be a positive example, a story of success and hope, as well as a story of the real-world limitations and goals that our work brings us. And, like all our research reports, it suggests how much more remains to be studied and understood.

APPENDIX 1

"The pretense operating in many schools is that teachers should treat all students the same, although numerous studies on teacher expectations have shown that race, class, and gender have considerable influence over the assumptions, conscious and unconscious, that teachers hold toward students" (Noguera, 1995, p. 203).

"One reason why the performance of females is not sustained through the secondary school years is because of their compliance, and consequent dependency, on the authority of the teacher" (Burton, 1989, p. 17).

"Culture is no longer viewed as static, one-dimensional, and uncontested, but as having multiple layers. This significant reconceptualization of multiculturalism interrogates the creation of difference within the context of history, culture, power, and ideology" (Schwartz. 1995, p. 636).

"We do not think of race and gender oppression in additive terms, an implication of phrases such as *double* and *triple jeopardy*. Rather, race, class, and gender are part of the whole fabric of experience for all groups, not just women and people of color" (Anderson and Collins, 1992, p. xii).

"The color-blind assumption can infer that student difference and defect are synonymous. It may result in missing opportunities to build on the lived experiences of many students. What appears fair may only exacerbate inequities" (Tate, 1995, p. 345).

"Speaking the language of critical pedagogy is neither necessary nor sufficient for building a diverse democratic community within the schools. This is the postmodernist trap that confuses a change in language with a change in the world" (Seixas, 1995, p. 436).

APPENDIX 2

"Roles may be reified in the same manner as institutions. The sector of self-consciousness that has been objectified in the role is then also apprehended as an inevitable fate, for which the individual may disclaim responsibility. The paradigmatic formula for this kind of reification is the statement, 'I have no choice in the matter, I have to act this way because of my position'" (Berger and Luckmann, 1967, p. 91).

"'Constructing' and 'introspecting' may not represent expectations for behavior that students in a particular district, school, or classroom are accustomed to meeting. If, instead, these students have routinely viewed appropriate actions associated with learning as memorizing information or as replicating problem solutions, then an important discrepancy exists between adults' reform expectations and students' daily enactment of the role" (Corbett and Wilson, 1995, p. 13).

"There is hurt in learning, and it is difficult to persuade someone to hurt himself. ... It is especially hard for adolescents, whose vulnerability and inexperience are attenuated. Getting them to pursue this often lacerating process of exposing that inexperience, and the errors it reaps, is a subtle, delicate business" (Sizer, 1984, p. 159).

"There is a certain point at which a person sees authority as an internal agent rather than as an external agent. At this point, truth is seen as eventuating from a personal perspective. It is here that one begins crossing the bridge from a submissive orientation to a position in which one's voice is a significant determiner of what one believes" (Cooney, 1994, p. 628).

"Students will construct, but we want their constructions to be guided by mathematical purposes, not be the need to figure out what teachers want or where they are headed" (Noddings, 1990, p. 16).

"Students who are intellectually autonomous in mathematics are aware of, and draw on, their own intellectual capabilities when making mathematical decisions and judgments as they participate in these practices. These students can be contrasted with those who are intellectually heteronomous and who rely on the pronouncements of an authority to know how to act appropriately" (Yackel and Cobb, 1996, p. 473).

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PhD Report 2

EXEMPLARY MATHEMATICS TEACHERS:
SUBJECT CONCEPTIONS AND INSTRUCTIONAL PRACTICES

Geoffrey Roulet
Queen's University, Kingston

> In fact, whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics. (Thom, 1973, p. 204)

The logical simplicity of René Thom's statement above, and general research linking beliefs to action (Abelson, 1979) encourage the assumption that there exists a natural connection between teachers' images of mathematics and their instructional practices related to the discipline. Theoretical models (Ernest, 1989) show teachers' mathematical beliefs, filtered somewhat by the opportunities and constraints presented by school contexts, leading to particular choices of instructional methods. This theory in turn implies, for those leading the present mathematics education reform movement, that "the single most compelling issue in improving school mathematics is to change the epistemology of mathematics in schools" (Romberg, 1992, p 433).

In fact, research investigating this presumed link, especially at the secondary school level, is not extensive. Studies exploring this issue, have generally been hampered by a lack of wide variability in both teachers' mathematical images and instructional methods. Most have found only narrow rule-based views of mathematics and corresponding traditional transmissive modes of instruction. This paper, and the EdD thesis (Roulet, 1998) of which it is a summary, add to past research by exploring connections between the the more open, richer conceptions of mathematics and the teaching practices of two exemplary Ontario high school mathematics teachers. In more formal terms the questions that guided the research were:

- What are the conceptions of mathematics held by exemplary secondary school mathematics teachers, those who are attempting to implement the reforms proposed by the National Council of Teachers of Mathematics [NCTM] (1989) and the Ontario Association for Mathematics Education [OAME] (1993)?

- To what extent are these teachers' instructional practices expressions of their subject images?

  and

- What are the struggles involved in these teachers' efforts to translate subject images into classroom practice?

LINKING CONCEPTIONS OF MATHEMATICS AND TEACHING PRACTICE: THEORY AND RESEARCH

The variety of categories and labels that have been used to collect together various conceptions of mathematics can be distilled to a scheme that identifies three groupings: instrumentalism, Platonism-formalism, and a problem-solving or social constructivist view. For instrumentalists, mathematics is a collection of unrelated facts, rules, and operations to be used in the pursuit of correct solutions to problems external to the subject. Platonists and formalists also view the discipline as a set of fixed laws and procedures, but the truth of these principles, rather than related to their utility, is assured by their connection to the underlying structure of the universe (Platonism) or to a solid axiomatic base and a well defined process of logical deduction (formalism).

In contrast to the "absolutist" (Ernest, 1991) positions above, is the "fallibilist" problem-solving (Lerman, 1983) or social constructivist (Bishop, 1985; Ernest, 1992) image of mathematics. This conception of the discipline begins with the view that mathematics is a human construct and thus subject to error and change. Mathematics is seen as an extension of natural language and, as a language, is acquired and developed through social interaction in our attempts to describe the world and solve the problems we meet.

The three philosophical positions have corresponding anticipated instructional styles. Teachers holding absolutist views of mathematics are expected to adopt teacher-centred transmissive modes of instruction, with instrumentalists focussing on careful execution of mathematical procedures and Platonist-formalists, in addition, emphasizing the reasons lying behind these processes. Teachers holding problem-solving or social constructivist views would see their classroom role as a facilitator, providing stimulating problems for investigation and building an environment in which pupils may discuss their emerging understandings. These anticipated connections have been explored in several classroom based empirical studies, but the picture is incomplete.

Studies in elementary grade classrooms (Heaton, 1992; Prawat, 1992; Putnam, 1992; Raymond, 1997; Remillard, 1992), in junior and senior secondary schools (Arsac, Balacheff and Mante, 1992; McGalliard, 1983), and at the college level (Ferrell, 1995) have located only instrumentalist conceptions of mathematics. Here, as expected, the teachers were found to employ transmissive modes of instruction. But, these results support only one half of the theoretical model; an apparent connection between absolutist subject images and teacher-centred direct instruction. These findings should not be surprising. The predominance of absolutist views of mathematics in the teaching profession and society at large (Steen, 1988) and the long standing tradition of teacher-directed transmissive instruction in school mathematics makes it relatively easy for those holding absolutist conceptions to translate their subject images into classroom action.

Teachers with well developed social constructivist philosophies of mathematics are not common and do not appear frequently in reported research. Dorgan (1994), at Grades 5 and 6, Thompson (1984), at Grades 7 and 8, and Kesler (1985), in senior secondary school, found both instrumentalist images and a few teachers with mixed or emerging problem-solving views. Kesler and Thompson's participating teachers with non-absolutist conceptions of mathematics did employ investigations in their lessons, but all of the teachers in Dorgan's study were found to use direct instruction. As Cooney's work (1985) with a beginning secondary school mathematics teacher shows, it is not easy to translate a problem-solving vision of the subject into lessons that demand significant student thought when pupils are quite content to experience mathematics as a set of rules. Only when teachers are involved in well supported projects designed to alter images of mathematics and encourage changed instructional practices (O'Brien, 1995; Philipp, Flores, Sowder and Schappelle, 1994; Wood, Cobb and Yackel, 1991) do we find social constructivist views and corresponding significant use of student investigations and conversation. Thus the link between fallibilist views of mathematics and non-traditional instruction has not been empirically well established. What is required is the identification of simultaneous occurrences of such images and practices, independent of officially supported mathematics education reform projects. The research project reported here had this aim.

EXPLORING SUBJECT IMAGES AND CLASSROOM PRACTICE

"Purposeful sampling" (Bogdan and Biklen, 1992) was employed in this research. Two teachers, Jonathan Ode and Randy Walker, whose non-traditional teaching I had experienced through their presentations at mathematics teachers' conferences and participation as Associate Teachers in the Queen's University BEd program, were invited to participate in the study.

Jonathan Ode is the senior of these two teachers, having considerable work and educational experience prior to beginning secondary school teaching. Jonathan began university studies in engineering but left after two years to enter the military where he received training in electronics technology. After five years of military service, Jonathan returned to university, completing a B.Sc. degree in mathematics and physics and continuing with four years of graduate school, receiving masters degrees in both philosophy and mathematics. Jonathan completed education courses through summer programs during his first three years of teaching. He has taught mathematics for 23 years and held the position of Assistant Department Head in large secondary schools (1500 plus students) in a suburban region of a large metropolitan centre. During his teaching career Jonathan has written articles in professional journals, made presentations at teachers' conferences, and participated in curriculum development activities.

Randy Walker has taught mathematics, physics and general science for 17 years and been the Assistant Head of the mathematics department in small (200 pupils) and medium sized (1000 pupils) secondary schools located in an industrial city with a population of 160,000. Randy began university studies in engineering but after two years switched to science, completing a degree with a major in mathematics. This was followed by five years employment in a variety of jobs, the longest lasting (3 years) as a management trainee in banking. Deciding that he wanted to be a teacher, Randy returned to university for a year and received a BEd degree in mathematics and physics teaching. Recent, personally directed exploration and study has lead to the development of resources for the high school teaching of fractal geometry and chaos. Using these materials, Randy has conducted teacher workshops and made conference presentations in a variety of Canadian and United States cities. Although both Jonathan and Randy had been observed to employ practices that reflected the messages of the mathematics education reform movement, neither was involved in any administratively supported curriculum change project.

Over a full semester (September to February), in two separate case studies, I spent a minimum of 25 hours in each teacher's classroom, observing at least 20 lessons with Grade 9 and senior secondary school classes. A sociological and epistemological perspective (Koehler and Grouws, 1992) guided the gathering of data on the teachers' instructional practices. That is, the focus was on the processes by which pupils constructed mathematical knowledge and the teacher's role in this, in: planning, creating supportive environments, setting tasks, grouping students, and orchestrating discourse. Whole class instruction portions of lessons were unobtrusively audio taped and illustrative segments later transcribed. Extensive field notes recording: classroom arrangements, questions posed, responses obtained, locations of interacting pupils, and interval times for various activities were kept. In addition, handouts used during the lessons and copies of representative student products were gathered.

The participants' images of mathematics and links to their teaching were investigated through twenty-one interviews (12 with Jonathan Ode, 9 with Randy Walker) ranging in length from a brief eight minutes up to almost an hour. To structure our conversations, but at the same time reduce the risk of distorting the teachers' voices, for each interview, the participants provided the focussing element: lesson plans, resource materials employed, a short personal essay on the nature of mathematics, a repertory grid comparing mathematics to other school subjects, and a concept map of the discipline. All interviews were audio taped and the resulting transcripts and related focussing elements were analysed for recurring themes.

With repeated passes through the data, pictures of Jonathan's and Randy's conceptions of mathematics were developed and dimensions of these linked to recurring features of their teaching. Incidents of compatibility and incompatibility between subject visions and practice were noted.

FROM SUBJECT CONCEPTION TO PRACTICE: TWO CASES

Jonathan Ode: Finding Strength in a Subject Image

Jonathan takes a strong constructivist position when asked to present his personal philosophy of mathematics. "I'm not a Platonist. I don't believe that there's a mathematics that we discover. I think we create it as we go along." This statement is not just simple rhetoric, for repeatedly in his writing, repertory grid, concept map, interviews, and in teaching Jonathan reveals a social constructivist stance.

For Jonathan, "mathematics has to be an active process," but it is, "doing with understanding." "By understanding I mean more than a passive understanding. That is, it has to be an understanding that can turn around and be communicated back." In his writing about the nature of the discipline, Jonathan makes it clear that mathematics is communication; communication of reasons along with answers. "A *necessary condition* for an activity being called mathematics is *the ability to explain the processes used* [emphasis in original]." In Jonathan's classroom, answers alone are not sufficient. Student work "doesn't take on any significance until they can explain how they got the answer." Whenever taking up homework Mr. Ode refused to provide answers, and instead moved to the back of the classroom from where he coached the pupils to collectively develop their own solutions and justifications.

Careful presentation of one's reasoning is important for Mr. Ode since, as he indicated with labels on his repertory grid, mathematics is neither "factual" nor "convergent." In Jonathan's view, "the concept of mathematical truth has changed drastically over the past fifty years. The correctness of the answer is now something that must be negotiated between players." In Mr. Ode's classes alternate interpretations of questions and resulting different solutions were encouraged. Often, multiple conflicting answers were left to stand until the class, through debate, decided which they would accept.

On Jonathan's concept map for mathematics, multiple connecting lines are labelled "problem solving." Problem solving is central to mathematics, but this means more than "going out and using formulas." "It's a matter of creating a structure that provides answers and then looking at the reasonableness of those answers and the structure itself." "Mathematics by its very nature is a higher level thinking process."

Mr. Ode frequently presented his pupils with "genuine problem solving" activities; situations where "they cannot immediately determine the answer without first investigating the problem, exploring various alternatives and deciding on the best strategy." To increase the opportunities for pupils to explain their thinking, collaborative group structures were employed for these lessons. In alternating "doer-listener" roles, pairs of students from the graduating year classes solved problems while thinking aloud in response to their partner's questions.

In Jonathan's vision of mathematics, "there are two prominent characteristics; patterning and problem solving." "Mathematics is the outgrowth of a number of human characteristics. We seem to have a need to simplify things of a complex nature, we need to organize things." The resulting mathematical models help satisfy our "desire to think beyond the immediate empirical data of this perceptual world." "We want to extend beyond what we have right in front of us and say well, if this situation varied what would happen....That's kind of part of our creative drive."

Jonathan sees parallels to the history of mathematics in the "learning development process." "You work your way up through the concrete into the conceptual level, so I think that it's natural to do things in a concrete way first." Mr. Ode followed his own advice in the Grade 9 introductory algebra unit, providing concrete referents for variables and expressions. The students used plastic cubes to build sequences of similar towers, each one layer larger than the previous example. The volumes and surface areas of the sample towers were recorded, patterns were noted, and mathematical expressions were developed to represent these quantities for a tower of the general *n*th level.

Jonathan's lessons, with their focus on investigations, problem solving, and student construction of multiple and sometimes conflicting solutions, made heavy intellectual demands of his pupils. About one-third of the way through the semester some students, especially those in his graduating year classes who were not earning the high grades needed for university admission, began to object to Mr. Ode's instructional style. Reports to parents received sympathetic responses, for these classes differed considerably from what they had experienced as school mathematics. When parental complaints reached the school office, the Vice-Principal visited Jonathan to discuss the concerns. Jonathan's competency was never questioned, but requests that he adopt a more traditional approach were made.

In the lessons I attended soon after this event, Mr. Ode led his senior classes through a mechanical Socratic development of the expression for the general term in the binomial expansion. But, this digression from his preferred approach did not last long. Within a month, I observed Jonathan reject the textbook's abstract formula approach to the topic of hypergeometric distribution, and help his classes develop the ideas through the generalization of patterns observed in progressively more complex specific examples.

Randy Walker: A Subject Image and Practice in Transition

Randy's image of mathematics does not fit conveniently into any of the theoretical category schemes. It is a philosophy in transition, showing features of each of the identified points of view. While constructing his repertory grid, struggling to group school subjects and give them descriptive labels, Randy acknowledged his amorphous and fluctuating conception of mathematics. "It's harder to say where mathematics exactly fits in. It's kind of between things, rather than being easily pinned down....I change my mind back and forth as I think about each description and I'm not sure what I'll think tomorrow."

Although recent reading related to fractal geometry and chaos has introduced Randy to debates concerning the nature of mathematics, he feels no pressing need to take a stand. "The discovering versus the inventing of mathematics—well I don't know where I stand on that. It's probably a little bit of both. There are a lot of ideas in philosophy that I don't find I have to take sides with." But, his new studies in fractal geometry have expanded Randy's subject image in one distinct direction. He now sees mathematics as developing through inductive processes. "Computer experiments now drive much of mathematics research. I probably never would have said that if I hadn't been involved in fractals over the last few years. I never would have had a notion of mathematics as being very experimental."

Randy took his reassignment to a new school and appointment as teacher for the enriched Grade 12 mathematics program as an opportunity to expand his efforts to "create a mathematics course where we would do much experimentation, and much less formal mathematics and the learning of techniques." Early in the semester, Mr. Walker expanded the topic of composition of functions to include explorations of recursion with the square and square root functions. The language of fractals: fixed points, attractors, and basins of attraction was introduced, and the class, using graphing calculators and a single classroom computer, conducted parallel investigations with a variety of quadratic functions.

Although Randy is excited to find that investigations have a place in his subject he does not see all of the discipline as having an experimental nature. When his school subjects repertory grid was expanded to identify particular mathematical sub-disciplines, fractal geometry was given the label "allows for expression and exploration", while calculus and algebra were described as "rigid." These two subject views, one for new mathematical topics and another for traditional course content, compete for expression in Randy's teaching and inform his responses to school curricular politics.

Conflict entered Randy's professional life soon after his efforts to expand the content of the enriched Grade 12 course. With the increased complexity of the work, students were finding the course more difficult than they had anticipated and test marks began to drop. In past years, admission to enriched mathematics courses had meant an opportunity to studying with the school's elite rather than increased academic demands. The competitive middle class parents of Randy's pupils communicated their desire to maintain this school tradition and in meetings with the Principal and Mathematics Department Head he was reminded that it was school policy to not assess enriched classes on any material beyond the course core. In the face of near universal rejection of new course content and methods, Randy's teaching increasingly followed the instrumentalist and formalist strands of his fractured subject image.

Three weeks after the reprimand concerning student evaluation, a sequence of observed lessons, focussing on the core Grade 12 topic of trigonometric functions, captured Randy's description of mathematics as "a body of knowledge, a set of rules, tools and techniques." Here, in highly teacher-centred lessons, Mr. Walker led his class through detailed descriptions of the curves of trigonometric functions and step-by-step rules for identifying and linking symbolic and graphical transformations of the sine and cosine functions. For Randy, traditional school mathematics "is the language of many technical subjects" and his students, with plans for university study in science and engineering, need practice with the methods of "concise" and "symbolic expression."

Randy's mixed and changing image of mathematics also involves the nature of proof. Admitting that "I may not really appreciate what mathematical proof is anymore because I used to think it was a lot simpler," Randy allows the possibility of valid arguments that do not employ formal deductive methods. "There are some graduations of convincibility where there seems to be overwhelming evidence; be it by virtue of a large number of examples, a complete lack of counter examples, or other strong evidence in the form of a logical construction." But in the end, this empirical approach to mathematics is not sufficient. "I think even many people in the field that are now using the computer as a tool know that's not enough. Ultimately to validate their work, their discoveries and computer experiments have to be expressed algebraically to fit into existing mathematics. Proof is still a necessity." A parallel approach was taken in class when trigonometric identities were discovered using graphing calculators, but then verified through formal deductive proofs. Mr. Walker presented mathematics as a formal system where, "we need to prove everything. We start with things that all of us agree are true and argue from there."

For Randy, applications are at the core of mathematics. "I can't really justify the existence of mathematics all by itself. There has to be a reason why the system exists." But, the need for techniques to solve practical problems is not the only driving force behind mathematics. Randy's concept map also contains a region titled "Desire for Understanding." "People outside of our discipline lump us in with technical subjects much more than we belong. I think mathematics is still considerably an art."

During the semester Randy struggled to find a balance between the instrumental and creative dimensions of mathematics. Acknowledging the school's administrative restrictions concerning pupil assessment in the enriched Grade 12 course, Randy presented mathematical investigations as bonus assignments. After a brief in-class introduction students were given the opportunity to investigate the ideas and submit their work for extra marks. Although the number of students responding to the invitations was not great Mr. Walker was excited by the work he received and regularly found after-school time to meet with those who wanted to discuss their explorations.

CONSTRUCTING A GUIDING ARGUMENT

Both Jonathan Ode and Randy Walker discovered that "to open up one's class so that students are pursuing problems whose outcomes cannot be easily foreseen is a hazardous business" (Black and Atkin, 1996, p. 132). Opposition came from pupils unwilling to make the extra intellectual efforts required to personally construct mathematical knowledge and these student complaints received support from parents who understood mathematics as a collection of precise algorithms and teaching as careful explanation of these procedures. Despite official curriculum documents (Ontario Ministry of Education, 1985) that supported their instructional methods, both Jonathan and Randy met administrative opposition to their teaching practices, and neither found support from their departmental colleagues.

With 17 or more years of teaching experience and reputations as competent professionals, Jonathan and Randy were never in any danger of dismissal from their jobs, but they still found a need to develop solid arguments for their classroom practices. The resulting private and internal debates were not motivated by desires to develop replies to those in opposition, but served the more personal need to rebuild self-confidence. Thus these arguments needed to satisfy the teachers' strong personal standards of logic and proof.

Jonathan's complex, integrated and consistent set of beliefs about mathematics provided him with a solid base for personally arguing for his classroom use of group work, student discussions, inductive reasoning from patterns, collaborative problem solving, and open-ended creative investigations. Randy's conception of mathematics, although complex, is less integrated and complete. His philosophy, a mixture of absolutist and fallibilist beliefs, did not permit the construction of a personally satisfying argument that would support action contravening school policy.

The Perry (1981) scheme of cognitive growth can be employed to compare Jonathan's and Randy's positions concerning mathematical epistemologies and teaching. Their statements in multiple interviews indicate that both are at least at the third stage in the scheme, Relativism. They realize that there exist multiple images of mathematics and a variety of approaches to teaching. Moreover, they understand that one can establish reasons for each position. Jonathan has in fact progressed beyond Relativism and has reached the stage of Commitment. He has made a choice of epistemologies and through reasoned thought come to a social constructivist position. From this stage of commitment he is able to take action and adopt practices that embody his philosophy. Randy appears to be in transition between Relativism and Commitment, and has not yet settled on an epistemological stance. Perry notes that periods of transition between stages, when one is trying to find a path, are unsettling. Arguments are not clear and there are choices to be made. It is difficult to select definitive actions and one may act in contradictory manners. Randy would appear to be at this point in forming his philosophy of mathematics.

Student, parent, and administrative opposition meant that Randy's efforts to bring open-ended investigations to the classroom were pushed to the margins of his practice, while Jonathan, when his teaching approach was questioned, was able to construct a personally satisfying argument that allowed him, with reason, to persist in his practice. Given the need for such a strong, rich image of mathematics to support the use of non-traditional teaching practices, it is not surprising that we find the pace of mathematics education reform painfully slow.

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PhD Report 3

RECOGNIZING AND SUPPORTING CHILDREN'S CONCEPTIONS OF NUMBER

Heather Kelleher
University of British Columbia

Seeing mathematical situations through the eyes of a child can be a challenge for adults. Once concepts become familiar and skills become automatic, it is difficult to reconstruct how a beginner might interpret the same situation. However, it is that very understanding of children's thinking and ways of knowing that can help parents and teachers provide the support children's need as they attempt to make sense of unfamiliar mathematical contexts.

One way to better understand children's quantitative thinking is to consider the different ways in which children approach and interpret number situations. These approaches, which are shaped by a child's personal reservoir of experience, interests and abilities, can be thought of as the lenses or filters through which the child interprets new number situations. As time goes on, children refine, extend and elaborate on their earliest conceptions about number, constructing alternatives and thereby enhancing their personal repertoire of approaches to dealing with number.

Research shows that young children conceptualize number in qualitatively different ways as they develop a progressively more sophisticated and complex understanding of number and its applications. Cobb (1990) described children's different "personal number contexts." Steffe and Cobb (1988) used children's progressively more sophisticated counting schemes to qualitatively differentiate children's mathematical activity. Confrey (1994) used the distinction between counting and splitting conceptualizations of number to account for qualitative differences in performance.

Building on this previous research, this article presents a framework for interpreting children's ways of making sense of number, and argues for the importance of supporting the construction of multiple meanings and contexts. It proposes that four qualitatively different conceptualizations of number constitute the frequently used lenses or filters through which young children interpret and approach number-related situations:

1. number as a global estimate,
2. number as counting,
3. number as pattern, and
4. number as grouping.

NUMBER AS A GLOBAL ESTIMATE

This first context which could be labeled "pre-number" is proposed to represent the initial and most prevalent conceptualization used by very young children up to roughly four or five years of age. It is characterized by a global approach to number situations without any focus on accuracy, but shows an understanding of the relationships or "big ideas" which provide the structure underlying quantitative situations. For example, very young children can use perceptual information to tell you who has more candies or cookies, and may give a number name or a quantifier such as "lots", while still being unable to tell how many in each group or tell how many more in one group than the other.

Research indicates that prior to the development of systematic quantification strategies, very young children begin to construct a number concept based on their experience with perceptual information in their environment. At the same time, with the support of older siblings, parents, and teachers, they begin to make their own sense of our socially and culturally shaped ways of using number. Over time children refine and elaborate their initial global and intuitive approach to number, and develop other more accurate and systematic ways of dealing with numbers and quantities. However, this early global approach remains available as an alternative, and may well be a preferred strategy in new and unfamiliar situations.

Resnick's (1983) protoquantitative schemas provide the conceptual basis for this approach. Resnick (1989) described these initial pre-number concepts as protoquantitative schemas which express quantity without numerical precision through perceptual rather than measurement processes. She described three such protoquantitative schemas: the comparison, the increase/decrease, and the part-whole schemas. According to Resnick, these three reasoning schemas constitute a major foundation for later mathematical development, and as language develops, pre-schoolers' implicit protoquantitative reasoning schemas combine with early counting knowledge to generate number concepts.

The perceptually grounded capacity to immediately recognize small groupings as specific numerosities is another quantification process that is available to children during this pre-number phase, and provides a second path to generating number concepts alongside counting. This capacity has been referred to as "subitizing" or "figural grouping" (von Glasersfeld, 1982). Very young children have been shown to be able to recognize specific grouping patterns and to assign number accordingly (Klahr & Wallace, 1973). In fact, the neo-nativist position (e.g., Carey and Gelman, 1991) posits that humans are pre-wired with a quantification capacity that even newborns have access to, and that experience is mapped onto this capacity over time. Once young children begin to apply these early intuitive notions in conjunction with reliable and systematic quantification strategies, they have moved beyond sole reliance on this first and earliest conceptualization for number. Children's reliance on their global, perceptually-based pre-number conceptualizations gradually is replaced by a preference for systematic quantification methods. However, pre-number thinking is still available to children as a fall-back position when the situation is beyond a child's number comfort zone.

NUMBER AS COUNTING

The linear order of unitary counting is generally recognized as children's initial basis for developing an understanding of whole number concepts and operations (Fuson, 1988; Gelman & Gallistel, 1978; Steffe and Cobb, 1988). Two qualitatively different unitary counting approaches, described here as early and later counting, are typical of children between the ages of 5 and 8.

An *early counting* approach is characterized by linear, unitary counting involving enumeration by ones and from one with direct modelling of the problem elements, rather than counting-on or internally representing any aspect of the problem. Piaget's ordinal counting, counting-all (e.g., Carpenter and Moser, 1984), and Steffe and Cobb's (1988) perceptual and figurative counting schemes provide the conceptual basis for this approach to number situations.

The concept of one-to-one correspondence plays a distinguishing role in differentiating this early counting level from the pre-number level of Number as a Global Estimate. Once students develop one-to-one correspondence, a reliable count-by-ones counting chain to match with objects, and the focus, coordination and inclination necessary to label and account for elements in a group, they begin to have a degree of accuracy to their counting and related number knowledge (Gelman and Gallistel, 1978).

A *later counting* approach is characterized by evidence of the ability to apply both the inclusion and order relations that are the basis of cardinal number (Piaget, 1952) as well as to conserve or recognize the invariance of number. Children approaching number situations from this later counting perspective make use of mental representations of number in order to count-on or back, and demonstrate the capacity to keep track of a double count for counting-on or back with tally. Fingers and other sorts of tally methods are widely used with this approach. Children making the shift between early and later counting approaches often construct a starting set with blocks, fingers, or words, but then disregard or mentally store the starting set and count on. At the most proficient end of the later counting approach, children confidently apply an internalized mental count to solve a wide range of problems. Just the slightest signs of counting might be apparent, such as head nods, eye or lip movements. Piaget's (1952) cardinal counting, Steffe and Cobb's (1988) initial number sequence, and Carpenter and Moser's (1984) counting-on provide different labels for describing this approach to number situations. Whether counting-all as in early counting or counting-on as in later counting, what characterizes this approach is a unitary, linear approach to number situations.

NUMBER AS PATTERN

Recognition of visual-spatial pattern-based conceptualizations of number has received relatively minimal attention in the literature as compared to the emphasis on counting. The importance of pattern approaches to quantification is that they provide an important link between children's early global-perceptual intuitions about number and the use of increasingly powerful grouping notions. There is some evidence that young children use non-counting approaches to number situations more frequently than is suggested in the literature (Kelleher, 1996). In that study, children frequently used perceptual estimates based on either the physical size of the starting set, visual-spatial patterns in arrays, or internalized number patterns and relationships rather than established count-by-one approaches.

This Number as Pattern approach may well reflect a different way of thinking, that of seeing number from a qualitative, global perspective rather than in a quantitative, linear way. This conceptualization is characterized by the use of visual-spatial pattern recognition as a means of making sense of a problem situation, in contrast to the use of linear, systematic, increasingly sophisticated unitary counting schemes. Von Glasersfeld's (1982) perceptual numbers, Confrey's (1994) splitting, and subitizing as described by Klahr and Wallace (1973) and others, provide the conceptual basis for this interpretation of number.

Number as Pattern conceptualizations (Kelleher, 1996) were often triggered by the perceptual characteristics of a concrete model, arrays in particular, and illustrated how younger children used pattern in unique and personal ways. For example, in a sharing task, six-year-old Cliff looked at twelve blocks he had arranged in a three by four array and immediately recognized without moving or counting, how many blocks two, three or four people would get.

In explaining to seven-year-old Chris's mother his interesting use of pattern in an interview, she reported how, as a pre-schooler, Chris developed a sense of number as pattern from working with Lego blocks. He would ask for specific blocks based on the dot pattern of the block, as in "I need an eight" when he wanted the block with two rows of four joiner dots. The patterns of Lego blocks provided him with a frame of reference for specific number patterns, and may well have had an impact on his construction of meaning for early number generally.

In some cases even a verbally presented problem without a corresponding physical model was interpreted by some children based on internalized patterns, as indicated by their explanations of their thinking. Explanations that made reference to intuitive, visualized patterns for number were usually more difficult to interpret than counting-based explanations, and usually were generated by the higher achievers.

Number as Pattern provides a complementary path alongside Number as Counting as a means of establishing systematic quantification procedures. In conjunction with counting, Number as Pattern is proposed to provide a second path between pre-number and grouping approaches to number situations. The most proficient children in the Kelleher study made use of both counting and pattern conceptualizations. In particular, visual-spatial pattern strengths appeared to characterize the performances of children who were the most proficient and creative in their approaches to number, while exclusive reliance on a counting conceptualization of number characterized the least proficient performances.

Mathematics education for young children has traditionally placed a heavy emphasis on counting approaches , often at the expense of pattern approaches. Confrey's (1994) work on the parallels between the construction of linear and exponential functions differentiated an alternative to counting. She distinguished her notion of splitting by its conceptual connections to the geometric transformation, similarity. Confrey argued that splitting, with its ties to partitioning, is an alternative basis for the construction of a number system and possesses strong explanatory potential for interpreting children's methods (p. 300). She described the split, an action of creating equal parts or copies of an original, as a primitive operation that is a precursor to a more adequate concept of ratio and proportion and subsequently to a multiplicative rate of change and the exponential and logarithmic functions. Confrey proposed that splitting and counting are complements that have their roots in the complementarity of geometry and arithmetic, with splitting structures producing geometric sequences and counting structures producing arithmetic sequences.

Counting models such as Steffe, von Glasersfeld, Richards, and Cobb (1983) do not adequately account for pattern conceptions of number that are holistic, symmetry oriented and visual-spatial in nature. Confrey's proposal of an exponential splitting (rather than linear counting) basis for number, provides a possible model to account for the global, spatially intuitive, pattern-based methods used by some students as a basis for interpreting number situations. These global methods appear to draw on the perceptually-based capacity to immediately recognize the numerosity of small groups (subitize) as well as the inclination to draw on qualitative visual-spatial approximation strengths. Both of these capacities appear to be independent of a linear counting conception for number, having closer ties to Resnick's (1983) pre-number proto-quantitative schemes and von Glasersfeld's (1982) figurative grouping.

NUMBER AS GROUPING

The fourth approach, Number as Grouping, may well represent the most powerful conceptualization for number in the realm of additive structures, and provides the connecting link with multiplicative structures. This approach to number situations for young children involves varying levels of use of a multi-unit conceptual framework, which enables taking into account the grouping patterns and relationships inherent in problem solving situations. Conceptually a grouping conceptualization of number requires a rich and flexible understanding of the additive composition of numbers including decomposition-recomposition, reversibility of thinking, fully elaborated part-whole understanding and an understanding of many-to-one correspondence.

Number as Grouping conceptualizations can be divided into two recognizable levels. *Early grouping* involves the non-systematic use of grouping concepts in the interpretation of multi-digit problem situations, involves multi-unit place value thinking, but does not include the capacity to consider a number situation simultaneously in both unitary and multi-unit terms. The conceptual basis for this early approach relates to Cobb and Wheatley's (1988) "ten as an abstract composite unit", as Kamii's (1986) second level of place value interpretation, and as Ross's (1989) stages three and four which are characterized by unreliable, inconsistent performance in coordinating ones and groupings of ten.

Established grouping describes an approach to number that highlights the multi-unit relationships inherent in problem situations and systematically applies grouping principles. This approach involves the simultaneous recognition of quantities both as a collection of ones and as various multi-unit groupings. This more sophisticated grouping approach corresponds to Kamii's (1986) third level of place value interpretation, and Cobb and Wheatley's (1988) ten as an iterable unit. This approach represents a fully operational grasp of the additive composition of number, which in turn can be considered an early form of multiplicative reasoning.

This description of four increasingly powerful conceptualizations of number (Global Estimate, Pattern, Counting, and Grouping) constitutes a synthesis and reworking of the literature into a practical framework for interpreting children's developing number sense. It recognizes that within the constraints of any particular view of number, children can demonstrate effective and creative ways to construct mathematical meaning.

SUPPORTING MULTIPLE CONCEPTUALIZATIONS OF NUMBER IN THE PRIMARY CLASSROOM

The four number schemes described above are proposed to constitute personal number contexts or lenses through which young children interpret number situations. With age and experience children expand their repertoire of possibilities to include multiple ways to conceptualize number. In light of this, then, what might be some implications for mathematics instruction in the primary grades?

Results (Kelleher, 1996) indicated that the most proficient performances were characterized by use of both pattern- and counting-based conceptualizations, while the least proficient performances were characterized by heavy reliance on only one of the two, in particular, unitary counting. One important implication might be that young children should be encouraged to develop the capacity to apply both pattern and counting conceptualizations for number, rather than developing an over-reliance on either of these two qualitatively different paths.

Emphasizing children's use of a pre-number global estimate approach as a means of developing pattern conceptualizations of number may well encourage use of multiple lenses for framing quantitative situations. One way could be to recognize, support, and build on children's intuitive, visual-spatial notions about number in more systematic and visible ways. The early introduction of grouping and sharing activities, emphasizing non-counting patterns first through subitizing small groups, then through finding these small groups within larger groups, might build on the perceptual grouping strengths of young children. Similarly, work with geometry and spatial relationships would enhance children's intuitive notions about number and its relationship to space. Informally dealing with area and tiling concepts earlier might be another approach. And finally, emphasizing the interpretation of graphs in terms of relative quantities and comparisons rather than solely as accurate representations of absolute values (e.g., discussing the big ideas behind area representations such as pie graphs) might tap into children's relative notions about numbers and quantities based on a wider range of perceptual information. Such examples of global rather than linear approaches to quantities would capitalize on children's intuitive understanding of quantity and provide a balance to the heavily unitary, linear, counting-based emphasis present in many primary classrooms.

Using these global conceptualizations of number in conjunction with linear, accurate counting-based interpretations of numbers and quantities would serve several purposes. It would provide a more inclusive approach to the understanding of number, accommodating and building on young children's different learning strengths and developmental levels. It would encourage the development of a wider range of strategies for dealing with number situations, thus placing counting strategies within a wider context of strategy options. It would place a higher priority on children's intuitive feel for number, and the meaning behind number and quantity situations. This would serve to support the development of estimation capacities as an important form of mathematical thinking. It may well serve to support the development of a multi-unit conceptual framework through early exposure and reliance on patterns and groupings. And finally, it would more directly connect with the patterns underlying our number system, thereby possibly providing more direct and reliable access to the development of multiplicative reasoning (Confrey, 1994).

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PhD Report 4

Learning to Teach Prospective Teachers: A Teacher Educator's View

Cynthia Nicol
University of British Columbia, BC

The realization of reform visions for teaching and learning mathematics poses new challenges for the mathematical and professional development of teachers. Designing classroom learning environments which promote mathematical inquiry and conceptual understanding requires, among other things, a developing knowledge of the content and discourse of mathematics, experiences of good mathematics teaching, knowledge of school mathematics, knowledge of students as learners of mathematics, and knowledge of mathematical pedagogy (NCTM, 1991). Yet such new visions of and requirements for mathematics education are, for the most part, a foreign experience for most students and teachers. Teachers themselves are successful graduates of mathematics schooling that has tended to focus on the learning and application of routine procedural skills.

How then might beginning and experienced teachers be expected to seriously consider teaching mathematics in ways they have not experienced either as learners or as teachers? How can mathematics teacher education programs provide support for beginning teachers to learn to teach differently, to learn to develop more powerful mathematics and mathematical thinking for their students, and to learn to teach for understanding? And assuming that learning to teach is a career-long process, how might prospective teachers be given support to develop and sustain a disposition for inquiry into practice that supports continuous investigation of subject matter content, pedagogical reasoning, and the social context of schooling?

As a beginning teacher educator these are some of the questions I ask myself as I consider what it might take to help prospective teachers be responsive to and critical of reform visions of mathematics education. In searching the literature, however, I have found few examples, studies, or stories of innovative mathematics methods courses or how reform visions might be interpreted by students and teachers. This highlights Cooney's (1994) claim for us that "research in teacher development in light of goals of the current mathematics education reform is in its infancy" (p. 613). There are also few stories of learning to teach as told *by* teacher educators *for* teacher educators or *about* teacher educators. This means that what teacher educators are learning in their practice is not often being communicated to their colleagues giving us limited understanding of the perspective and learning of mathematics teacher educators. Yet, what teacher educators learn and how they manage their problems of practice are important spaces of inquiry—ones that have implications for the improvement of mathematics teacher education and for the professional development of teacher educators.

To address the challenges of reform visions, I sought to design, in collaboration with my colleagues a mathematics methods course that might help prospective teachers learn to consider teaching differently—and hopefully better—than that which they had most likely experienced as students. I use my experiences teaching this course as a site for inquiry into what might be entailed in learning to teach prospective teachers to teach mathematics. A main focus is an examination of the pedagogical deliberations made, the tensions, dilemmas, and challenges experienced, and the insights gained during a mathematics methods course that I taught to prospective elementary teachers. Highlighted are my own and prospective teachers' actions and re-actions to the course events together with my claims of resistance and their calls for listening. It is my intention to open up possibilities for discussion about teacher educators' learning and the ways in which teacher educators might be supported in their learning to teach prospective teachers.

THEORETICAL CONSIDERATIONS

Learning Mathematics

Mathematics for many people is commonly associated with being able to get the right answer quickly without the need for creativity, understanding, or inquiry (NRC,1989; Schoenfeld, 1985, 1989a; Stodolsky 1988). Such views are not necessarily intended goals of instruction, but they can be developed through years of experience in classrooms in which doing mathematics has meant remembering and using collections of facts and procedures, teaching mathematics has meant the direct transmission of established knowledge, and mathematical truth has been established by the teacher or the textbook. Beliefs about how to learn mathematics, how to do it, and how it should be taught have been acquired through years of listening, observing, and practicing mathematics and have become "both a cause and a logical consequence of the ways in which knowledge is regarded in school mathematics lessons" (Lampert, 1990 p. 32). Students' and teachers' sense of what mathematics is all about is shaped by the culture of school mathematics.

Mathematics as it is practiced in school is very different from the way in which it is practiced by mathematically literate adults or by research mathematicians. A potentially useful way of describing the various communities for which mathematics is the subject is offered by Richards (1991). He suggests that research math, inquiry math, journal math, and school math, represent four different cultures, each with different assumptions and goals for what it means to know and do mathematics. *Research math* is the spoken discourse of the professional mathematician. *Inquiry math* is that used by mathematically literate adults as they participate in mathematical discussions and debates, ask mathematical questions and think critically about the use of mathematics in the popular media. *Journal math* is the formal communication of mathematics and represents the product of the mathematical activity rather than the activity itself. Richard's fourth community for which mathematics is the subject is *school math* and it is defined as the discourse of the standard mathematics classroom. In this context "what is learned is useful for solving habitual, unreflective, arithmetic problems ... this discourse does not produce mathematics, or mathematical discussions, but rather a type of 'number talk' that is driven by computation" (p. 16).

Richards highlights the similarities between inquiry math and research math noticing that they are both based on a logic of discovery. In contrast, he suggests that journal math and school math tend to focus on the product of the mathematical activity rather than the ways in which the mathematics was constructed. In this way journal math and school math are grounded in a logic of reconstruction. As Richards notes, school mathematics with its focus and emphasis on the product of mathematical activity does not help students become familiar with the activity of doing mathematics for themselves, it does not help students make mathematics more personally meaningful, and it does not help students learn about what it means to engage in mathematical discussions as mathematically literate individuals. "Obviously," says Richards "*school math* is the wrong math to be teaching students" (1991, p. 16).

Learning to Teach Mathematics

How might prospective teachers learn to teach mathematical inquiry when they themselves have such vivid memories and experiences of Richards' (1991) school math. How might they "unlearn" (Ball 1990a), their school mathematics in order to open up possibilities for considering alternatives? A pedagogy of inquiry, notes Richards, involves providing opportunities for students to engage in a discourse of inquiry mathematics. Prospective teachers need opportunities to participate in such a discourse. They need to know what it feels like to learn mathematics from an inquiry perspective—to unlearn their school mathematics. This involves engaging prospective teachers in questioning what they know and believe to be true about mathematics, and about teaching and learning it. It involves helping them make the familiar unfamiliar and make problematic what they see as the way things necessarily are.

A mathematics methods course involves learning what mathematics teaching is, could be, and ought to be. However, a mathematics methods course, as Ball (1990a) reminds us, is as much about learning pedagogy as it is about learning mathematics. Prospective teachers need opportunities to participate in a discourse of mathematical inquiry. But they also need opportunities to participate in a discourse of pedagogical inquiry—a discourse which values inquiry into teaching mathematics, learning mathematics, and learning to teach mathematics. For teacher educators it means creating spaces of possibilities for prospective teachers to develop both mathematical and pedagogical reflective, critical stances. But what might this entail? How might prospective teachers be invited to participate in a discourse of inquiry, what might be the focus of inquiry, and what settings might be used to promote such inquiry?

Developing a Pedagogy of Inquiry for Teacher Education

If we parallel Richards' idea of a pedagogy of inquiry for teaching mathematics with a pedagogy of inquiry for learning to teach mathematics, we can move from the typical conception of methods courses as places for the development of technical, how-to skills, to methods courses as places of inquiry—both mathematical and pedagogical inquiry. Methods courses which focus on presenting teachers with ideal teaching methods based on a synthesis of educational theory do not represent the complexity and uncertainty of teaching. Such methods courses would as Doyle (1990) suggests give the impressions that there are definite answers to educational problems and that teaching is a matter of following and applying practical principles as sets of procedures in practice. And such methods courses tend to separate a study or interrogation of the subject matter from a study of the pedagogy.

A pedagogy of inquiry, on the other hand, would begin to shift the emphasis of learning to teach from a focus on only limiting instruction to the best teaching methods to an emphasis on discussion, critique, and investigation of pedagogical problems that arise in the context of practice. Such a pedagogy of teacher education parallels the mathematical pedagogy of reform efforts as it "involves watching and listening to the learner, helping the learner to identify and articulate assumptions, and bringing new perspectives to bear on the interpretation and solution of the problem" (Heaton and Lampert, 1994, p. 47-48). This would be in line with a "new pedagogy of teacher education" suggested by Lampert, Heaton, and Ball (1994).

CONCEPTUAL FRAMEWORK FOR THE MATHEMATIC COURSE

In order to provide prospective elementary teachers with opportunities to experience mathematics differently from their previous experiences and in order to help prospective teachers consider how they might teach mathematics differently, my colleagues Ann Anderson, Sandra Crespo, Klaus Hoeschmann[1], and I designed a mathematics methods course with a number of goals in mind. These included our desire to open up spaces for the investigation of reform visions of mathematics education so that prospective teachers might question their ownideas about the nature of mathematics and what might be entailed in teaching and learning it. We were also interested in providing opportunities for prospective teachers to investigate teaching—their own teaching and that of others—with a desire to help them perceive themselves as resources for their own learning of mathematics and of teaching mathematics.

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[1] Ann Anderson is an Associate Professor and Sandra Crespo was a doctoral candidate in mathematics eduation at the University of British Columbia (UBC), while Klaus Hoeschmann is a professor in the mathematics department UBC.

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Program Context

The course we taught was a required methods course for prospective elementary teachers in the Faculty of Education's two-year teacher education program—a program which extends over two academic years of two terms. Teacher candidates who are admitted to this program have at least 3 years of course work through the Faculties of Arts and Sciences and are required to have at least one university level mathematics course offered through the Mathematics Department as one of the program admission requirements.

The teacher education program at this university is structured to offer general and foundation courses in curriculum and instruction and in developmental psychology during the first term of the year; methods courses are offered during the second term; an extended 13-week practicum the first term of the second year; and course work focusing on the social context of schooling is offered during the second term of the final year. During the first year the mathematics methods course is one of eight methods courses prospective teachers take during the second term[2]. There is typically no field experience associated with the methods courses, however, prospective teachers do attend a program 2-week practicum mid-way through the course.

Course Context

The mathematics methods course that my colleagues and I taught, and around which this study is based, met twice a week for 1.5 hours each Wednesday and Friday morning over an 11 week period from January to the end of March 1995. Our goal was to design an inquiry-oriented course—one that began to blur the boundaries between learning mathematics in a mathematics course, learning to teach mathematics in a methods course, and learning to teach in a practicum setting. We wanted to engage prospective teachers in an investigation of mathematics, teaching, learning and learning to teach through interactions with students. And we wanted to use these interactions with students as springboards for mathematical and pedagogical inquiry. To meet our goals we designed the course with some special features: working with students, using a content theme, and practicing a pedagogy of investigation for ourselves and for prospective teachers.

Working with students. First we introduced a field-based experience to our course. To address the pitfalls of experience that Feiman-Nemser and Buchmann (1986) speak of we sought to provide prospective teachers with opportunities to think about and discuss the complexities of teaching mathematics through collaborative settings. We provided prospective teachers opportunities to work with students either through a letter writing exchange project with Grade 4 students (Crespo, 1998) or through working with small groups of Grade 6/7 students at an elementary school close to the university (Nicol, 1997). Our Wednesday classes were spent as a whole group while our Friday classes were spent working with students in either of these projects. Our intent was for prospective teachers' interactions with students to form the context through which we might as a class and individually investigate mathematical and pedagogical problems.

Teaching through the theme of multiplicative thinking. We designed the content of the course around the theme of multiplicative thinking through three topic domains of the mathematics curriculum (Ministry of Education, 1987), number sense and operations, size and shape, and data and chance.

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[2] The eight methods courses taken concurrently focus on subject matter disciplines of: art, reading, science, mathematics, social studies, physical education, language arts, and music.

Multiplicative thinking, as the theme for the course allowed us to dwell deeply in a few topics rather than cursively over many issues of mathematics teaching and learning. The inherent difficulty of multiplicative thinking for both adults and school children (Behr, Harel, Post and Lesh, 1992; Geer, 1992; Hoechsmann, 1996) makes the study of it a challenging and fruitful site for prospective teachers to engage in mathematical and pedagogical investigations.

A pedagogy of investigation. A major project for prospective teachers was an investigation of their own teaching through their interactions with students. This project was designed to be a major feature of the course through which much of our discussion and investigation of teaching occurred. The interactions with students provided opportunities for prospective teachers to think about, investigate, interpret and discuss the circumstances experienced within a collaborative setting. I was responsible for working with the group of fourteen prospective teachers who had chosen to work with students in the school for their major project (I called this the Collaborative Inquiry group).

The Collaborative Inquiry group met during our Wednesday classes to plan and prepare for their Friday visits to the Grade 6/7 math classroom. Prospective teachers worked in pairs to prepare and teach their small group of Grade 6/7 students. We chose a common problem to pose to the students so that we might have a shared context through to discuss our experiences. Prospective teachers were, however, encouraged to extend, adapt, and revise these problems to meet the needs and interests of their students. After an hour of working with the students we met as a group in the school for a debriefing meeting—a place for prospective teachers to share their insights and experiences in working with the students. Prospective teachers were involved in investigating their own teaching as they audio taped their small group interactions, collected samples of student work, and took notes during our debriefing meetings. They were also expected to keep a journal of their developing and evolving ideas related to their experiences in the course and in particular in working with the students.

It is this group of prospective teachers, the Collaborative Inquiry group, who are the focus of my investigation and whose experiences form much of the data for the analysis of my work as a teacher educator. The focus of this paper is a description and critique of my learning to teach prospective teachers. In particular I study what I found problematic, that is, the problems of practice that elicited surprise, perplexity, uncertainty, or doubt (Dewey, 1933; Schön, 1987) and the pedagogical possibilities that might be offered teacher educators in the design of inquiry-oriented methods courses.

RESEARCH METHOD

I, like Gore (1991), am interested in working to align "the pedagogy of our talk" with "the pedagogy we talk about." As prospective teachers were researching their interactions with students I was interested in researching my interactions with prospective teachers, and the pedagogical deliberations that arose for me as I attempted to engage prospective teachers in both mathematical and pedagogical inquiry through their working with students. Richardson (1996) calls for teacher educators to investigate their practice both through practical inquiry and formal research. This study is an attempt to do this.

To describe and critique my learning to teach prospective teachers this study is divided into two parts. One part documents my teaching at various turning points or critical incidents (Tripp, 1993) in the course. I recount what I found problematic, how I resolved those problems, what I was learning, and my reflections on teaching as it occurred during the course. Another part analyzes these events from a perspective one year later as I continue to learn from my practice through my retrospective study of it. This paper focuses on a critical incident or turning point that occurred during the third and fourth weeks of the course.

Participants

This is a study of my learning to teach prospective teachers but it necessarily involves the participation of others: prospective teachers, the students prospective teachers worked with, and teachers of these students. Fourteen of 34 prospective teachers enrolled in the course requested to work with the Grade 6/7 students in the school, that is, requested the Collaborative Inquiry project as their major project. All of these prospective teachers accepted the invitation to participate in the research (i.e., gave permission to be within range of the video camera and to allow their coursework to be used as data for the study). All 14 came to the course with undergraduate degrees; their major areas of study included English, history, geography, international relations, human kinetics, economics, sociology, psychology, and chemistry and physics. Half of those in the Collaborative Inquiry group had not taken any university mathematics courses beyond high school until they learned that it was a requirement for admission into the teacher education program.

Fifty Grade 6/7 students also participated in the research with their teachers Jake and Sabrina.[3] I first met Jake and Sabrina in the spring of 1994 as part of their involvement in a school-district action research group. They were interested in making changes in their mathematics instruction and were wondering how they might make mathematics more conceptually oriented for their students as well as how they might accommodate diverse learners in mathematics.

In terms of my own background, I came to teaching this methods course with seven years experience of teaching Grade 8-12 mathematics in a school in a small coastal fishing village. It was not until my third year of graduate study that I became involved in teaching prospective teachers. This was the first course I had taught which attempted to embed field experiences within it and to use these experiences as springboards for inquiry into issues of teaching and learning mathematics.

Data Sources

Data were gathered from a variety of sources. The main research tools used for investigating my practice were my journal, prospective teachers' journals and course work, audio-taped recordings of our (the instructors') collaborative planning sessions, and video-taped recordings of the methods course class sessions both at the university and at the school. I first began writing a journal for this study in June of 1994 and used the journal not only to describe and record events related to the design and enactment of the course but also as a thinking tool. For example, I used the journal to try out various responses to prospective teachers that might impact or influence their learning. Prospective teachers kept a journal as a course requirement and wrote of their investigations as learners in the course and as teachers investigating their interactions with students. With prospective teachers' consent these journals were photocopied, together with prospective teachers' course assignments, at the end of the course.

Most of our planning meetings as course instructors were audio taped and transcribed. For those planning meetings that sometimes occurred spontaneously and were not recorded, I wrote notes in my journal of the events and the nature of our discussions and then shared them through e-mail communications. In addition to these data sources, semi-structured interviews (Mishler, 1986) were conducted with nine prospective teachers near the beginning of the course and at the end of the course. As well, the planning sessions with the collaborating teachers in the school were audio taped.

Data Analysis

There were three main phases of data analysis in this study. The first phase occurred directly following my classroom teaching as I wrote in my journal reflecting on the course activities at the time the course was taught. This is a reflection on the action present (Schön, 1987) and parallels Richardson's

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[3] Pseudonymns are used for all prospective teachers, students, and classroom teachers.

(1996) practical inquiry. A second phase occurred a year later as I analyzed my own journal, prospective teachers' journals, planning meeting records, and classroom video tape records for what I found problematic, what I was attending to, and what I found surprising, difficult, and challenging. This analysis provided various themes and patterns in what I was attending to and to my actions and reactions to the challenges faced over the course. During this second phase I constructed a critical incident map of the turning points or well-remembered events of the course and used this as a framework in helping decide which stories to tell of my experiences and what they might be stories of. The stories constructed are considered as part of the interpretive process, they are as Carter (1993) notes, "not merely raw data from which to construct interpretation but products of a fundamentally interpretive process" (p. 9). A third phase of analysis occurred in drawing upon various theoretical frameworks to help make sense of the course events with respect to what a beginning teacher educator might find problematic and the pedagogical possibilities that might be offered in designing on campus inquiry-oriented experiences for prospective teachers.

I communicate much of my experiences in the form of stories and I do this as a deliberate strategy for a number of reasons. First stories bring the reader closer to the problems and challenges I faced and help the reader experience vicariously the course events. Second, stories allow us to communicate,the specific and complex nature of teaching about teaching mathematics. And third, stories make our experiences accessible to others and provide spaces of invitation for us to deliberate with each other on the complexities of teaching prospective elementary teachers to teach mathematics.

What Might a Teacher Educator Find Problematic?

> I identify a problem when some of educational values are denied in my practice. (McNiff, 1993, p. 7)

What does it mean to say that aspects of practice are problematic? Dewey's (1933) reflective inquiry and Schön's (1987) design inquiry suggest that inquiry begins with situations that are problematic, that is, situations that are confusing, uncertain, or conflicted. It is the situation that arises that is perplexing, unexpected, and surprising that sets the context and frame for the focus of what might be considered as problematic. Much of the teaching experienced in this year was new to me. I embarked on this teaching with images of what I thought teaching prospective teachers might be; how those images would be played out from class-to-class and moment-to-moment were less clear. What I found problematic involved both situations of creating and designing plans for teaching as well as enacting them; situations which Schön (1987) would refer to as nonroutine problems as opposed to familiar routine problems. But, at times during the planning and teaching of the course and more so a year later in the analysis of my practice, I sought to problematize that which I did not, at the time, take as problematic. In addition, there are situations in which it was not until this study of my practice a year later that I was able to recognize some of the "living contradictions" that I hold between my beliefs and my actions of practice (McNiff, 1993).

The problematic aspects of practice I discuss here are those which I found to be most salient throughout my teaching. I have organized what I found problematic around the five areas of choosing and using worthwhile pedagogical tasks, generating and sustaining pedagogical discussion, attending to prospective teachers, interweaving investigations of mathematics and pedagogy, and developing a community of inquiry. Although I have organized these topics discretely, they are not discrete topics. For example, embedded in what I found problematic in choosing and using worthwhile pedagogical tasks are also difficulties in working with the interplay between mathematical and pedagogical inquiry. Here I discuss three areas which highlight aspects involved in teaching prospective teachers (see Nicol, 1997 for further discussion of these and other areas).

Choosing and Using Worthwhile Pedagogical Problems

In searching for appropriate contexts through which to explore, reason, and investigate both mathematics and pedagogy, I sought tasks that were 1) inviting, motivating, and engaging; 2) challenging in terms of helping prospective teachers re-interpret their past experiences and confront their assumptions and beliefs ofmathematics, teaching, and learning; 3) genuine in the sense that they portrayed through their engagement and investigation the complexities and uncertainties of teaching mathematics for understanding; and 4) illuminating in terms of providing insights into prospective teachers' thinking. However it was my commitment to these goals that often made choosing and using tasks problematic. Two tensions that are salient for me during the course: 1) the tension between wanting problems to be both challenging and supportive; and 2) the tension between posing problems that are both familiar and at the same time unfamiliar to prospective teachers.

Challenging and supportive. By engaging prospective teachers in problems that challenged their underlying assumptions about teaching mathematics, we risked disabling rather than enabling their confidence in themselves to learn, teach, and learn to teach mathematics. Yet, without situations which encouraged prospective teachers to examine thoughtfully their preconceptions and what they assumed, their current thinking would remain unchallenged. Some prospective teachers responded to this with active or passive resistance, others by returning to more familiar routines of teaching mathematics to substantiate old beliefs, and for others it meant seriously questioning themselves as both learners and teachers of mathematics. The challenge for me lies in choosing and using problems in ways that challenge prospective teachers' thinking and respects their beliefs, but also provide the support for them to take the risk to investigate and question their previously accepted ideas for teaching mathematics.

Familiar and unfamiliar. Related to my dilemma of choosing and posing problems that were both challenging and supportive is the relationship between activities that are familiar and unfamiliar. Using Schön's (1987) development of the learning predicament, I cannot tell prospective teachers what it is about teaching mathematics they need to know in ways that they will first understand, because such understanding and experience cannot be available as lists of procedures or steps to follow. Yet, it is difficult for prospective teachers to gain experience as learners of reform-oriented teaching without some understanding of what it is they are to learn. Activities chosen and how they are used will therefore be, to some extent, unfamiliar to prospective teachers. For prospective teachers to want to engage with the problems requires that the problems be somewhat familiar.

The challenges and difficulties associated with my choosing and using appropriate worthwhile problems are similar to that presented by Romagnano (1994), Ball (1990b) and Ball and Wilson (1996). Although Romagnano, Ball and Wilson discuss the challenges, difficulties, and uncertainties of selecting and presenting appropriate instructional representations for Grade 3 and Grade 9 students in mathematics and social studies, their claims about the inherent difficulties are similar to my own. The results of my study indicate that considerations of instructional representations through choosing and using various tasks are not unique to elementary and secondary teachers, but include also to teacher educators.

Interweaving Investigations of Mathematics and Pedagogy

Shulman (1986) identifies the intersection between subject matter knowledge and teaching as pedagogical content knowledge.[4] Many of the previous studies on subject matter knowledge in teaching have investigated novices' development of this kind of knowledge, novices who have had the opportunity to learn subject matter followed by opportunities to learn how to use or transform their subject matter knowledge (e.g., Grossman, 1990; Wilson, Shulman, & Richert, 1987). In our course, however, prospective teachers were simultaneously trying to un-learn and re-learn a subject while learning to teach, presenting me with a challenge of how to interweave the study of mathematics and teaching. When I began the course I considered the investigation of both as an integral part of the course and the design of the activities we posed. I recognized that if the course were to make a difference for prospective teachers in developing their knowledge and beliefs about mathematics, students, and pedagogy, it would need to provide opportunities for them to both experience learning the mathematics they would eventually be expected to teach in ways aligned with reform visions as well as to experience learning to teach it in such ways. How I might coordinate the investigation of the mathematics and pedagogy was less certain for me. I wondered how I might shift smoothly between the contexts, when to intervene in a mathematical discussion to address a pedagogical issue, and when to change directions to investigate a mathematical issue.

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[4] Shulman (1986) defines pedagogical content knowledge as one of several components of the broader construct of strategic knowledge—a theoretical construct for the knowledge needed to make moment-to-moment decisions in situations where general principles of practice do not seem to work.

Developing a Community of Inquiry

The idea of a classroom as a learning community has been explored by many researchers (e.g., Ball, 1993; Lampert, 1990; Langrall, Thornton, Jones, and Malone, 1996; Schoenfeld, 1987, 1989b; Schwab, 1975; Wilcox, Schram, Lappan, and Lanier, 1991). Creating a community of learners with shared responsibility for learning is likely to provide an environment in which learners can freely take emotiohal and intellectual risks. However, in our course over the short time we had together I found that, although developing a community of inquiry was a goal, it was one that was not easily attained. Aspects of developing collaboration and inquiry that I found problematic included: 1) developing relationships with and among prospective teachers in which prospective teachers felt comfortable sharing and examining their own and their peersF practice; 2) collaborating with my colleagues in a situation where my voice seemed to be heard over the others; 3) creating opportunities for the group to investigate the experiences of anotherFs practice to further the understandings of the group as a whole and deciding which ideas are most appropriate to open up for the group to investigate; 4) maintaining my role as both a teacher and a learner, that is, maintaining both my credibility and authenticity as a teacher; 5) creating collaboration and investigation of both mathematics and pedagogy; and 6) balancing support for prospective teachers while at the same time challenging their ideas and thinking in learning to teach.

What I found problematic in terms of developing a stance of inquiry in a collaborative setting grew from my commitments to design authentic activities for prospective teachers to investigate and learn about teaching and to respect prospective teachersF thinking in terms of their ability to investigate for themselves aspects of their practice. If I were to present to prospective teachers pedagogical techniques developed by expert teachers or researchers, prospective teachers might not learn to see themselves as capable of investigating their teaching. While seeking to represent learning to teach as more than the learning of various pedagogical techniques through the presentation of teaching activities I found it a challenge to establish a learning environment conducive to critique and inquiry.

The work of teaching prospective teachers to teach mathematics is complex and filled with dilemma and tensions of teaching about a kind of teaching that is itself uncertain and undetermined. Rosaen and Wilson (1995) note, "the work of teaching is tough enough without problematizing it at yet another level" (p. 52). I agree. Teacher educators if seen as both teachers and learners, need the support of various resources including curriculum materials and other teacher educators if they are to continue to learn about their practice and to make a difference for prospective teachers' learning and the students these prospective teachers will eventually teach.

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APPENDICES

APPENDIX A

WORKING GROUPS AT EACH ANNUAL MEETING

- 1977 Queen's University, Kingston, Ontario
  - Teacher Education programmes
  - Undergraduate mathematics programmes and prospective teachers
  - Research and mathematics education
  - Learning and teaching mathematics

- 1978 Queen's University, Kingston, Ontario
  - Mathematics courses for prospective elementary teachers
  - Mathematization
  - Research in mathematics education

- 1979 Queen's University, Kingston, Ontario
  - Ratio and proportion: a study of a mathematical concept
  - Minicalculators in the mathematics classroom
  - Is there a mathematical method?
  - Topics suitable for mathematics courses for elementary teachers

- 1980 Université Laval, Québec, Québec
  - The teaching of calculus and analysis
  - Applications of mathematics for high school students
  - Geometry in the elementary and junior high school curriculum
  - The diagnosis and remediation of common mathematical errors

- 1981 University of Alberta, Edmonton, Alberta
  - Research and the classroom
  - Computer education for teachers
  - Issues in the teaching of calculus
  - Revitalising mathematics in teacher education courses

- 1982 Queen's University, Kingston, Ontario
  - The influence of computer science on undergraduate mathematics education
  - Applications of research in mathematics education to teacher training programmes
  - Problem solving in the curriculum

- 1983 University of British Columbia, Vancouver, British Columbia
  - Developing statistical thinking
  - Training in diagnosis and remediation of teachers
  - Mathematics and language
  - The influence of computer science on the mathematics curriculum

- 1984 University of Waterloo, Waterloo, Ontario
  - Logo and the mathematics curriculum
  - The impact of research and technology on school algebra
  - Epistemology and mathematics
  - Visual thinking in mathematics

1985 Université Laval, Québec, Québec
    Lessons from research about students' errors
    Logo activities for the high school
    Impact of symbolic manipulation software on the teaching of calculus

1986 Memorial University of Newfoundland, St, John's, Newfoundland
    The role of feelings in mathematics
    The problem of rigour in mathematics teaching
    Microcomputers in teacher education
    The role of microcomputers in developing statistical thinking

1987 Queen's University, Kingston, Ontario
    Methods courses for secondary teacher education
    The problem of formal reasoning in undergraduate programmes
    Small group work in the mathematics classroom

1988 University of Manitoba, Winnipeg, Manitoba
    Teacher education: what could it be
    Natural learning and mathematics
    Using software for geometrical investigations
    A study of the remedial teaching of mathematics

1989 Brock University, St. Catharines, Ontario
    Using computers to investigate work with teachers
    Computers in the undergraduate mathematics curriculum
    Natural language and mathematical language
    Research strategies for pupils' conceptions in mathematics

1990 Simon Fraser University, Vancouver, British Columbia
    Reading and writing in the mathematics classroom
    The NCTM "Standards" and Canadian reality
    Explanatory models of children's mathematics
    Chaos and fractal geometry for high school students

1991 University of New Brunswick, Fredericton, New Brunswick
    Fractal geometry in the curriculum
    Socio-cultural aspects of mathematics
    Technology and understanding mathematics
    Constructivism: implications for teacher education in mathematics

1992 ICME-7, Université Laval, Québec, Québec

1993 York University, Toronto, Ontario
    Research in undergraduate teaching and learning of mathematics
    New ideas in assessment
    Computers in the classroom: mathematical and social implications
    Gender and mathematics
    Training pre-service teachers for creating mathematical communities in the classroom

- 1994 University of Regina, Regina, Saskatchewan
  - Theories of mathematics education
  - Preservice mathematics teachers as pruposeful learners: issues of enculturation
  - Popularizing mathematics
- 1995 University of Western Ontario, London, Ontario
  - Anatomy and authority in the design and conduct of learning activity
  - Expanding the conversation: trying to talk about what our theories don't talk about
  - Factors affecting the transition from high school to university mathematics
  - Geometric proofs and knowledge without axioms
- 1996 Mount Saint Vincent University, Halifax, Nova Scotia
  - Teacher education: challenges, opportunities and innovations
  - Formation à l'enseignement des mathématiques au secondaire: nouvelles perspectives et défis
  - What is dynamic algebra?
  - The role of proof in post-secondary education
- 1997 Lakehead University, Thunder Bay, Ontario
  - Awareness and Expression of Generality in Teaching Mathematics
  - Communicating Mathematics
  - The Crisis in School Mathematics Content
- 1998 University of British Columbia, Vancouver, British Columbia
  - Assessing Mathematical Thinking
  - From Theory to Observational Data (and Back Again)
  - Bringing Ethnomathematics Into the Classroom in a Meaningful Way
  - Mathematical Software for the Undergraduate Curriculum

APPENDIX B

PLENARY LECTURES

1977	A.J. Coleman	The objectives of mathematics education
	C. Gaulin	Innovations in teacher education programmes
	T.E. Kieren	The state of research in mathematics education
1978	G.R. Rising	The mathematician's contribution to curriculum development
	A.I. Weinzweig	The mathematician's contribution to pedagogy
1979	J. Agassi	The Lakatosian revolution*
	J.A. Easley	Formal and informal research methods and the cultural status of school mathematics*
1980	C. Cattegno	Reflections on forty years of thinking about the teaching of mathematics
	D. Hawkins	Understanding understanding mathematics
1981	K. Iverson	Mathematics and computers
	J. Kilpatrick	The reasonable effectiveness of research in mathematics education*
1982	P.J. Davis	Towards a philosophy of compuation*
	G. Vergnaud	Cognitive and developmental psychology and research in mathematics education*
1983	S.I. Brown	The nature of problem generation and the mathematics curriculum
	P.J. Hilton	The nature of mathematics today and implications for mathematics teaching*
1984	A.J. Bishop	The social construction of meaning: a significant development for mathematics education?*
	L. Henkin	Linguistic aspects of mathematics and mathematics instruction
1985	H. Bauersfeld	Contributions to a fundamental theory of mathematics learning and teaching
	H.O. Pollak	On the relation between the applications of mathematics and the teaching of mathematics
1986	R. Finney	Professional applications of undergraduate mathematics
	A.H. Schoenfeld	Confessions of an accidental theorist*
1987	P. Nesher	Formulating instructional theory: the role of students' misconceptions*
	H.S. Wilf	The calculator with a college education
1988	C. Keitel	Mathematics education and technology*
	L.A. Steen	All one system

1989	N. Balacheff	Teaching mathematical proof: the relevance and complexity of a social approach
	D. Schattsneider	Geometry is alive and well
1990	U. D'Ambrosio	Values in mathematics education*
	A. Sierpinska	On understanding mathematics
1991	J .J. Kaput	Mathematics and technology: multiple visions of multiple futures
	C. Laborde	Approches théoriques et méthodologiques des recherches Francaises en didactique des mathématiques
1992	ICME-7	
1993	G.G. Joseph	What is a square root? A study of geometrical representation in different mathematical traditions
	J Confrey	Forging a revised theory of intellectual development Piaget, Vygotsky and beyond*
1994	A. Sfard	Understanding = Doing + Seeing ?
	K. Devlin	Mathematics for the twent-first century
1995	M. Artigue	The role of epistemological analysis in a didactic approach to the phenomenon of mathematics learning and teaching
	K. Millett	Teaching and making certain it counts
1996	C. Hoyles	Beyond the classroom: The curriculum as a key factor in students' approaches to proof
	D. Henderson	Alive mathematical reasoning
1997	R. Borassi	What does it really mean to teach mathematics through inquiry?
	P. Taylor	The high school math curriculum
	T. Kieren	Triple embodiment: Studies of mathematical understanding-in-interaction in my work and in the work of CMESG/GCEDM
1998	J. Mason	Structure of Attention in Teaching Mathematics
	K. Heinrich	Communicating Mathematics or Mathematics Storytelling

*These lectures, some in a revised form, were subsequently published in the journal *For the Learning of Mathematics*.

APPENDIX C

PROCEEDINGS OF ANNUAL MEETINGS OF CMESG/GCEDM

Past proceedings of the Study Group have been deposited in the ERIC documentation system with call numbers as follows:

Proceedings of the 1980 Annual Meeting	ED 204120
Proceedings of the 1981 Annual Meeting	ED 234988
Proceedings of the 1982 Annual Meeting	ED 234989
Proceedings of the 1983 Annual Meeting	ED 243653
Proceedings of the 1984 Annual Meeting	ED 257640
Proceedings of the 1985 Annual Meeting	ED 277573
Proceedings of the 1986 Annual Meeting	ED 297966
Proceedings of the 1987 Annual Meeting	ED 295842
Proceedings of the 1988 Annual Meeting	ED 306259
Proceedings of the 1989 Annual Meeting	ED 319606
Proceedings of the 1990 Annual Meeting	ED 344746
Proceedings of the 1991 Annual Meeting	ED 350161
Proceedings of the 1993 Annual Meeting	ED 407243
Proceedings of the 1994 Annual Meeting	ED 407242
Proceedings of the 1995 Annual Meeting	ED 407241
Proceedings of the 1996 Annual Meeting	Not yet assigned*
Proceedings of the 1997 Annual Meeting	ED 423116

There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.

*These *Proceedings* have been submitted to ERIC.

APPENDIX D

LIST OF PARTICIPANTS AT THE 1998 MEETING

Ann Anderson
CUST, Faculty of Education
University of British Columbia
2125 Main Mall
Vancouver, BC V6T 1Z4
Tel: (604)822-5298
Fax: (604)822-4714
e-mail: ann.anderson@ubc.ca

Pat Barnson
69 Willowmeaded Crescent
Winnipeg, MB R2M 4P4
Tel: (204) 253-9750
E-mail: umbarns3@cc.umanitoba.ca

Nadine Bednarz
CIRADE
Université du Québec à Montréal
C.P. 8888, Succursale A
Montréal, QC H3C 3P8
TEL: (514) 987-3012
E-mail: Descamps-Bednarz.nadine@uqam.ca

Stephen Campbell
Department of Education
University of California, Irvine
2001 Berkeley Place
Irvine, CA 92697-5500 USA
Tel: (949) 824-7736
Fax: (949) 824-2965
E-mail: sencael@uci.edu

Jon Carrodus
1365 Oakwood Cres.
North Vancouver, BC V7P 1L6
Tel: (604) 987-2595
Fax: (604) 987-5691
E-mail: carrodus@sfu.ca

Olive Chapman
Faculty of Education
2500 University Drive NW
University of Calgary
Calgary, AB T2N 1N4
Tel: (403) 220-5640
Fax: (403) 282-8479
E-mail: chapman@acs.ucalgary.ca

Rina Cohen
Dept. of C.T.L., OISE/UT
252 Bloor St. W.
Toronto, ON M5S 1V6
Tel: (416) 923-6641 ext 2477
Fax: (416) 926-4744
E-mail: rcohen@oise.utoronto.ca

Alan Cooper
Langara College
100 W. 49th Ave.
Vancouver, BC V5Y 2Z6
Tel: (604) 323-5676
Fax: (604) 323-5555
E-mail: acooper@langara.bc.ca

Mary L. Crowley
Department of Education
Mount St. Vincent University
Halifax, N.S. B3M 2J6
Tel: (902) 457-6304
Fax: (902) 457-4911
E-mail: mary.crowley@msvu.ca

Brent Davis
Faculty of Education
York University
4700 Keele St.
North York, ON M3J 1P3
Tel: (905) 729-8950
Fax: (416) 736-5409
E-mail: bdavis@edu.yorku.ca

A. J. (Sandy) Dawson
Faculty of Education
Simon Fraser University
Burnaby, BC V5A 1S6
Tel: (604) 291-5969
Fax: (604) 291-5021
E-mail: dawson@sfu.ca

Lydia De Santos
Kwantlen University College
Richmond, BC V7E 6M2
Tel: (604) 277-2528
Fax: (604) 599-2716

Astrid Defence
2085 Melrose Ave.
Montreal, PQ H4A 2R6
Tel: Res. (514) 483-0268
E-mail: astrid@vax2.concordia.ca

Luc Demers
Département de mathématiques
Université d'Ottawa
Ottawa, ON K1N 6N5
Tel: (613) 562-5800 ext 3537
E-mail: ldemers@oreo.uottawa.ca

Jean Dionne
Dép. d'enseignement et d'apprentissage
Faculté des sciences de l'éducation
Université Laval
Québec, QC G1K 7P4
Tel: (418) 656-2131 ext 3977
Fax: (418) 656-2905
E-mail: jean.dionne@fse.ulaval.ca

Malgorzata Dubiel
Dept. of Math & Statistics
Simon Fraser University
Burnaby, BC V5A 1S6
Tel: (604) 291-3800
Fax: (604) 291-4947
E-mail: dubiel@cs.sfu.ca

Don Eastman
Dept. of Math/Computer Science
Brandon University
Brandon, MB R7A 6A9
Tel: (204) 727-9696
E-mail: eastman@brandonu.ca

Gary Flewelling
Box 417
Arthur, ON N0G 1A0
Tel: (519) 848-2342
E-mail: flew@sentex.net

Doug Franks
Faculty of Education
Nipissing University
Box 5002
North Bay, ON P1B 8L7
Tel: (705) 474-3461, Ext.4457
Fax: (705) 474-1947
E-mail: dougf@mail.unipissing.ca

Sandra Frid
Dept. of Curriculum Studies
University of New England
Armidale, NSW
Australia 2351
Tel: 612 677 35063
Fax: 612 677 35078
E-mail: sfrid@metz.une.edu.au

George Gadanidis
2145 Whitfield Road
Port Perry, ON L9L 1B2
Tel: (905) 666-6391
E-mail: gadanidis@sympatico.ca

Kgomotso Garekwe
Dept. of Curriculum: Math & Sciences
Faculty of Education
University of Manitoba
Winnipeg, MB R3T 2N2
Tel: (204) 275-7924
Fax: (204) 474-7550
E-mail: umgarek@cc.umanitoba.ca

Linda Gattuso
Département de Mathématiques
UQAM
CP8888 Succ. Centre Ville
Montréal, QC H3C 3P8
Tel. (514) 987-3000 Ext. 6714
E-mail: gattuso.linda@uqam.ca

Claude Gaulin
F.S.E. 734
Université Laval
Québec, QC G1K 7P4
Tel: (418) 656-7570
Fax: (418) 656-2000
E-mail: cgaulin@fse.ulaval.ca

Susan Gerofsky
#218 - 2918 Slocan
Vancouver, BC V5M 4S7
Tel: Res. (604) 438-0218
Fax: (604) 291-3203
E-mail: gerofsky@sfu.ca

Florence Glanfield
Dept. of Curriculum Studies
College of Education
University of Saskatchewan
28 Campus Drive
Saskatoon SA S7N 0X1
Tel: (306) 966-7564
Fax: (306) 966-7658
E-mail: florence.glanfield@usask.ca

Frédéric Gourdeau
Dept. de Mathématiques et de Statisque
Université Laval
Québec, PQ G1K 7P4
Fax: (418) 656-2817
E-mail: frederic.gourdeau@mat.ulaval.ca

José Guzmán-Hernández
UQAM (CIRADE)
case postale 8888 succursale A
Montréal, QC H3C 3P8
Tel: (514) 987-3000 poste 703
Fax: (514) 987-4636
E-mail: c3121@er.uqam.ca

Pamela Hagen
1034 Gatenbury Street
Port Moody, BC V3H 2P2
Tel: (604) 931-7870
Fax: (604) 931-8899
E-mail: phagen@direct.ca

Gila Hanna
Dept. of MECA, O.I.S.E./U.T.
252 Bloor St. W.
Toronto, ON M5S 1V6
Tel: (416)923-6641 (Ext.2474)
Fax: (416) 926-4744
E-mail: ghanna@oise.utoronto.ca

Kathy Heinrich
Dept. of Mathematics and Statistics
Simon Fraser University
Burnaby, BC V5A 1S6
Tel: (604) 291-3925
Fax: (604) 291-5876
E-mail: heinrich@cs.sfu.ca

Dave Hewitt
School of Education
University of Birmingham
Edghaston, Birmingham
B15 2TT, UK
Tel: 44 121 414 4824
Fax: 44 121 414 4865
E-mail: d.p.hewitt@bham.ac.uk

William Higginson
A235 McArthur Hall
Queen's University
Kingston, ON K7L 3N6
Tel: (613) 545-6000, Ext.7403
Fax: (613) 545-6580
E-mail: higginsw@educ.queensu.ca

Joel Hillel
Dept. of Mathematics and Statistics
Concordia University
7141 Sherbrooke St. W.
Montréal, QC H4B 1R6
Tel: (514) 848-3235
Fax: (514) 848-2831
E-mail: jhillel@alcor.concordia.ca

Bernard R. Hodgson
Dept. de mathématiques et de statistique
Université Laval
Québec, QC G1K 7P4
Tel: (418) 656-2131 poste 2975
Fax: (418) 656-2817
E-mail: bhodgson@mat.ulaval.ca

Klaus Hoechsmann
Department of Mathematics
University of British Columbia
Vancouver, BC V6J 1Z2
Tel: Res. (604) 822-5458
Fax: (604) 822-6074
E-mail: hoek@math.ubc.ca

Martin Hoffman
33 Cambridge Place
Brooklyn, NY 11238
Tel: (718) 789-7857
Fax: (718) 997-3132
E-mail: hoffman@qcvaxa.acc.qc.edu

Lars Jansson
Faculty of Education
University of Manitoba
Winnipeg, MB R3T 2N2
Tel: (204) 474-9039
Fax: (204) 474-7550
E-mail: jansson@cc.umanitoba.ca

Ann Kajander
88 S. High St.
Thunder Bay, ON
P7B 3K4
Tel: (807) 343-8469
Fax: (807) 346-8788
E-mail: akajande@flash.lakeheadu.ca

Heather Kelleher
1200 Hilary Place
North Vancouver, BC V7J 3N6
Tel: (604) 985-9878
Fax: (604) 822-4714
E-mail: heather.kelleher@ubc.ca

Carolyn Kieran
Dept. of Mathematics
Université du Québec à Montréal
C.P. 8888, Succ. Centre-ville
Montréal, QC H3C 3P8
Tel: (514) 987-3000 ext. 7793
Fax: (514) 987-8935
E-mail: kieran.carolyn@uqam.ca

Thomas E. Kieren
Dept. of Secondary Education, #341 Ed. South
University of Alberta
Edmonton, AB T6G 2G5
Tel: (403) 492-3673
Fax: (403) 492-9402
E-mail: tom.kieren@ualberta.ca

Maria Klawe
Dean of Science
University of British Columbia
Vancouver, BC V6T 1Z2
Tel: (604) 822-0677
E-mail: klawe@interchg.ubc.ca

Lloyd Lawrence
5-66 Edmonton St.
Winnipeg, MB R3C 1P7
Tel: (204) 474-8299
E-mail: umlawre0@cc.umanitoba.ca

Lesley Lee
5324 Waverly
Montréal, QC H2T 2X7
Tel: (514) 987-3000 ext 0301
E-mail: r36114@er.uqam.ca

Dave Lidstone
Mathematics and Statistics Dept.
Langara College
100 West 49th Ave.
Vancouver, BC V5Y 2Z6
Tel: (604) 323-5795
Fax: (604) 323-5555
E-mail: dlidston@langara.bc.ca

John Mason
CME, Open University
Milton Keynes MK7 6AA UK
Tel: 44-1908-653550
Fax: 44-1908-652140
E-mail: j.h.mason@open.ac.uk

Ralph Mason
Faculty of Education
University of Manitoba
Winnipeg, MB R3T 2N2
Tel: (204) 474-9086
Fax: (204) 474-7550
E-mail: masonrt@ms.umanitoba.ca

Bruce McAskill
456 Dressler Road
Victoria, BC V9C 2E7
Tel: (250) 356-7678
Fax: (250) 356-2316
E-mail: bruce.mcaskill@gems4.gov.bc.ca

Douglas McDougall
Dept. of Curriculum Teaching and Learning
OISE/UT, 252 Bloor St. W.
Toronto, ON M5S 1V6
Tel: (416) 923-6641 Ext. 2249
Fax: (416) 926-4744
E-mail: dmcdougall@oise.utoronto.ca

David McLaughlin
Kwantlen University College
12666 72nd Ave.
Surrey, BC V3W 0A4
Tel: Res. (604) 599-2201
Fax: (604) 599-2279
E-mail: davidm@kwantlen.ubc.ca

Jim Mennie
209-2170 West 3rd Ave.
Vancouver, BC V6K 1L1
Tel: (604) 733-1882
Fax: (604) 733-1882
E-mail: jmennie@direct.ca

Joyce Mgombelo
Dept. of Secondary Education
341 Education South
University of Alberta
Edmonton, AB T6G 2G5
Tel: (403) 492-3760
Fax: (403) 492-9402
E-mail: mgombelo@gpu.srv.ualberta.ca

Michael Monagan
Dept. of Mathematics and Statistics
Simon Fraser University
Burnaby, BC V5A 1S6
Tel: (604) 291-4279
E-mail: monagan@cecm.sfu.ca

Eric Muller
Deptpartment of Mathematics
Brock University
St. Catharines, ON L2S 3A1
Tel: (905) 688-5550 Ext. 3297
Fax: (905) 682-9020
E-mail: emuller@spartan.ac.brocku.ca

Roberta Mura
Faculté des sciences de l'education
Université Laval
Ste. Foy, QC G1K 7P4
Tel: (418) 656-2131 Ext. 2145
Fax: (418) 656-2905
E-mail: roberta.mura@fse.ulaval.ca

Cynthia Nichol
Faculty of Education (CUST)
2125 Main Mall
Vancouver, BC V6T 1Z4
Tel: Res. (604) 279-8471
Fax: (604) 822-4714
E-mail: ccnicol@unixg.ubc.ca

Thomas O'Shea
Faculty of Education
Simon Fraser University
Burnaby, BC V5A 1S6
Tel: (604) 291-4517
Fax: (604) 291-5882
E-mail: oshea@sfu.ca

Eileen Phillips
3669 W. 18th Ave.
Vancouver, BC V6S 1B3
Tel: (604) 221-2637
Fax: (604) 713-5448
E-mail: eileen@cs.ubc.ca

David Pimm
116L Erickson Hall
Michigan State University
E. Lansing, MI 48824-1034
Tel: (517) 353-3736
E-mail: pimm@pilot.msu.edu

Susan Pirie
Dept. of Curriculum Studies
2125 Main Mall
University of British Columbia
Vancouver, BC V6T 1Z4
Tel: (604) 822-5206
Fax: (604) 822-4714
E-mail: susan.pirie@ubc.ca

Yvonne Pothier
Deparment of Education
Mount St. Vincent University
Halifax, NS B3M 2J6
Tel: (902) 457-6183
Fax: (902) 475-4911
E-mail: yvonne.pothier@msvu.ca

Medhat H. Rahim
Faculty of Education
Lakehead University
Thunder Bay, ON P7B 5E1
TeL: (807) 343-8053
Fax: (807) 346-7771
E-mail: medhat.rahim@lakeheadu.ca

Graham Rankin
Dept. of Mathematics
Kwantlen University College
Surrey, BC V3W 2M8
Tel: (604) 599-2203
Fax: (604) 599-2279
E-mail: graham@kwantlen.bc.ca

David Reid
Seminary House Room 206
School of Education
Acadia University
Wolfville, NS B0P 1X0
Tel: (902) 585-1393
Fax: (902) 585-1071
E-mail: david.reid@acadiau.ca

David Robitaille
Faculty of Education
University of British Columbia
Vancouver, BC V6T 1Z4
Tel: (604) 822-5529
E-mail: david.robitaille@ubc.ca

Pat Rogers
Faculty of Education
York University
4700 Keele St.
Toronto, ON M3J 1P3
Tel: (416) 736-5002
Fax: (416) 736-5913
E-mail: progers@edu.yorku.ca

Geoffrey Roulet
Faculty of Education
Queen's University
Kingston, ON K7L 3N6
Tel: (613) 545-6000 Ext.4935
Fax: (613) 545-6584
E-mail: rouletg@educ.queensu.ca

James Sherrill
Faculty of Education
University of British Columbia
2075 Westbrook Place
Vancouver, BC V6T 1Z5

Christine Shiu
Centre for Mathematics Education
Open University
Milton Keynes, U.K. MK7 6AA
Tel: 44-1908-652323
Fax: 44-1908-652140
E-mail: c.m.shiu@open.ac.uk

Elaine Simmt
341 Education South
Dept. of Secondary Education
University of Alberta
Edmonton, AB T6G 2G5
Tel: (403) 492-0881
Fax: (403) 492-9402
E-mail: elaine.simmt@ualberta.ca

Shannon Sookochoff
10915 - 127th St.
Edmonton, AB T5M 0S8
Tel: (403) 455-6084
E-mail: sas3@gpu.srv.ualberta.ca

Susan Stuart
Nipissing University
100 College Dr., Box 5002
North Bay, ON P1B 8L7
Tel: (705) 474-3461 Ext.4236
Fax: (705) 474-1947
E-mail: sues@einstein.unipissing.ca

Peter D. Taylor
Dept. of Mathematics & Statistics
Queen's University
Kingston, ON K7L 3N6
Tel: (613) 545-2434
Fax: (613) 545-2964
E-mail: taylorp@post.queensu.ca

Jo Towers
3651 West 26th Avenue
Vancouver, BC V6S 1P2
Tel: Res.(604) 222-1085
E-mail: jmtowers@unixg.ubc.ca

Jana Trgalová
Dept. of Mathematics and Statistics
Concordia University
7141 Sherbrooke St. West
Montreal, PQ H4B 1R6
Tel: (514) 848-3245
Fax: (514) 848-2831
E-mail: trgal@vax2.concordia.ca

David Wheeler
205-1230 Haro Street
Vancouver, BC V6E 4J9
Tel: Res.(604) 683-7846
E-mail: david_wheeler@sfu.ca

Appendix D

Vicki Zack
St. George's School
5822 Einstein Ave.
Montreal, PQ H4W 2X6
Tel: (514) 486-5214
Fax: (514) 486-5638
E-mail: edvz@musica.mcgill.ca

Rina Zazkis
Faculty of Education
Simon Fraser University
Burnaby, BC V5A 1S6
Tel: (604) 291-3662
Fax: (604) 291-3203
E-mail: rina_zazkis@sfu.ca